Assignment 13 – All 2 parts – Math 412

Optional!

Textbook exercises:[1]

Section 7.5: 1, 2

Other exercises:

(1) (WI) Fourth isomorphism theorem for groups. Let G be a group and let $N \trianglelefteq G$ be a normal subgroup.

(a) For a subgroup $H \leq G$ with $H \supseteq N$, let $H/N = \{hN : h \in H\}$. Show that H/N is a subgroup of G/N.

(b) Let $\pi : G \to G/N$ denote the natural homomorphism $\pi(g) = gN$. Given a subgroup $\overline{H} \leq G/N$, let $H := \pi^{-1}(\overline{H})$. Show that $H \leq G$, $H \supseteq N$, and that $\overline{H} = H/N$.

(c) Conversely, for $H \leq G$, show that $H = \pi^{-1}(H/N)$.

(d) Parts (b) and (c) give an order-preserving bijection

$$\{\text{Subgroups } H \leq G : H \supseteq N\} \leftrightarrow \left\{\text{Subgroups } \overline{H} \leq G/N\right\}$$

$$\begin{aligned} \{\text{Subgroups } H \leq G : H \supseteq N\} &\longleftrightarrow \left\{\text{Subgroups } \overline{H} \leq G/N\right\} \\ H &\longmapsto H/N \\ \pi^{-1}(\overline{H}) &\longmapsfrom \overline{H} \end{aligned}$$

Now, show that this bijection preserves the property of being normal, i.e. show that H is a normal subgroup of G if and only if H/N is a normal subgroup of G/N.

(2) Let $k, n \in \mathbf{Z}_{\geq 1}$ and let $\sigma \in S_n$ be a k-cycle. Show that $\#\sigma = k$.

(3) (WI) Let $k_1, \ldots, k_m \in \mathbf{Z}_{\geq 1}$ and let $\sigma_1, \ldots, \sigma_m \in S_n$ be disjoint cycles with σ_i a k_i-cycle. Show that the order of $\sigma_1\sigma_2 \cdots \sigma_m$ is $\operatorname{lcm}(k_1, k_2, \ldots, k_m)$.

(4) Write out the elements of A_3 and A_4 as products of disjoint cycles.

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*

(5) Write the element

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 7 & 6 & 5 & 4 & 2 \end{pmatrix}$$

as a product of disjoint cycles. What is $\operatorname{sgn}(\sigma)$?