Assignment 2 – Part 1 – Math 412

Textbook exercises:[1]

Section 1.3: 10 (WI), 17

Section 2.1: 6

Other exercises:

(1) (WI) Prime factorization, gcds, and lcms. Let a and b be non-zero integers and let $p_1, \ldots, p_k > 0$ be the positive primes that divide a or b (or both!). Write

$$a = up_1^{e_1}p_2^{e_2}\cdots p_k^{e_k} \quad \text{and} \quad b = vp_1^{f_1}p_2^{f_2}\cdots p_k^{f_k}$$

with $u, v \in \{\pm 1\}$ and $e_i, f_i \geq 0$ (recall from class that u, v, and the e_i and f_i are unique).

(a) Show that $a \mid b$ if and only if $e_i \leq f_i$ for all $i = 1, 2, \ldots, k$.

(b) Show that

$$\gcd(a, b) = p_1^{g_1}p_2^{g_2}\cdots p_k^{g_k}$$

where $g_i = \min(e_i, f_i)$.

(c) The *least common multiple* of a and b is the smallest positive integer that is divisible by both a and b. It is denoted $\operatorname{lcm}(a, b)$ or simply $[a, b]$ (like the gcd is sometimes denoted simply (a, b)). Show that

$$\operatorname{lcm}(a, b) = p_1^{h_1}p_2^{h_2}\cdots p_k^{h_k}$$

where $h_i = \max(e_i, f_i)$.

(d) Show that $ab = (a, b)[a, b]$.

(2) Suppose p is a prime number and $a \in \mathbf{Z}$.

(a) Let $n \in \mathbf{Z}_{\geq 0}$. Show that if $p \mid a^n$, then $p^n \mid a^n$.

(b) Let $e \in \mathbf{Z}_{\geq 0}$. We say that p^e *exactly divides* a (written $p^e \mid\mid a$) if $p^e \mid a$ and $p^{e+1} \nmid a$. Let $b \in \mathbf{Z}$. Let $b \in \mathbf{Z}$ and suppose $p^e \mid\mid a$ and $p^f \mid\mid b$. Show that $p^{e+f} \mid\mid ab$.

(c) Suppose $p^e \mid\mid a$ and $p^f \mid\mid b$. Show $p^{\min(e,f)} \mid b$. Show by example that it can happen that $p^{\min(e,f)}$ does not exactly divide $a + b$.

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*