Assignment 4 – All 2 parts – Math 412

Due in class: Thursday, Sept. 26, 2019

Textbook exercises:[1]

None yet.

Other exercises:

(1) Let $R = \mathbf{Q}$ be the field of rational numbers and let p be a prime number. Recall that every element r in $\mathbf{Q}$ can be written as $r = \frac{a}{b}$ with $a, b \in \mathbf{Z}$ and $\gcd(a, b) = 1$; we then say that $\frac{a}{b}$ is a reduced representation of r. Let

$$\mathbf{Z}_{(p)} := \left\{ \frac{a}{b} \in \mathbf{Q} : \frac{a}{b} \text{ is a reduced representation and } p \nmid b \right\},$$

i.e. the set of rational numbers whose denominator is not divisible by p.

(a) Show that $\mathbf{Z}_{(p)}$ is a subring of $\mathbf{Q}$.

(b) Show that if $n \in \mathbf{Z}$ is not divisible by p, then n is a unit in $\mathbf{Z}_{(p)}$.

(c) Show that p is not a unit in $\mathbf{Z}_{(p)}$.

(2) Let $S = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbf{Z}\}$. Show that this is a subring of $\mathbf{R}$. It is denoted $\mathbf{Z}[\sqrt[3]{2}]$.

(3) (WI) Let R be a ring and let

$$Z(R) := \{z \in R : zr = rz \text{ for all } r \in R\}.$$

(a) Show that $Z(R) \leq R$. It is called the *centre of* R.

(b) Let R be the quaternions $\mathbf{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbf{R}\}$ and let $S = \{a + bi \in \mathbf{H}\}$. Show that S is a commutative subring of $\mathbf{H}$, but there are elements in $\mathbf{H}$ that do not commute with elements of S.

(4) Let R be a commutative ring and let $n \in \mathbf{Z}_{\geq 1}$. Consider the ring $M_n(R)$ and its subset

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*

S consisting of the diagonal matrices whose diagonal entries are all equal, i.e.

$$S = \left\{ \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & a & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & a & 0 \\ 0 & \dots & 0 & 0 & a \end{pmatrix} : a \in R \right\} \subseteq M_n(R).$$

These are called the *scalar matrices*.

(a) Show that S is a subring of the centre of $M_n(R)$.

(b) In fact, $S = Z(M_n(R))$. For simplicity, show this when $n = 2$. (Hint: start by supposing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the centre and compare $AE^{(1,1)}$ with $E^{(1,1)}A$; then compare $AE^{(1,2)}$ with $E^{(1,2)}A$, where

$$E^{(1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E^{(1,2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(the general $n \times n$ case can be handled with generalizations of these matrices that one would denote $E^{(k,\ell)}$)).

(5) (WI) Let R_1 and R_2 be nonzero rings and let $R = R_1 \times R_2$ be their direct product. Let

$$R_1' := \{(r_1, 0) \in R_1 \times R_2\}.$$

Then, you can see that R_1' is a ring (you don't need to show this).

(a) Is R_1' a subring of R?

(b) Suppose $R_1 = R_2$ and let $R' = \{(r, r) \in R_1 \times R_1\}$. Is R' a subring of $R_1 \times R_1$?

(6) (WI) Let R be a commutative ring and let $\mathcal{F}$ denote the ring $\mathrm{Func}(R, R)$ of R-valued functions on R. Given a polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[X]$$

we can define a function $F_f : R \to R$ by

$$F_f(r) = a_0 + a_1 r + \cdots + a_n r^n.$$

Let $\mathrm{Poly}(R,R) \subseteq \mathrm{Func}(R,R)$ be the functions $F : R \to R$ that are equal to F_f for some $f(x) \in R[x]$, i.e. $\mathrm{Poly}(R,R)$ is the set of functions $F : R \to R$ given by a formula

$$F(r) = a_0 + a_1 r + \cdots + a_n r^n,$$

with $a_i \in R$ and $n \in \mathbf{Z}_{\geq 0}$. We call $\mathrm{Poly}(R,R)$ the set of *polynomial functions* for R to itself.

(a) Show that $\mathrm{Poly}(R,R)$ is a subring of $\mathrm{Func}(R,R)$.

(b) For $R = \mathbf{Z}/2\mathbf{Z}$, show that every R-valued function on R is a polynomial function. (Hint: you can just find, by hand, a polynomial that gives every function.)

(c) For $R = \mathbf{Z}/2\mathbf{Z}$, consider the two polynomials $f_1(x) = x$ and $f_2(x) = x^2$. These are not equal as polynomials, but show that they give the same polynomial functions on $\mathbf{Z}/2\mathbf{Z}$, i.e. that $F_{f_1} = F_{f_2}$. (Remark: the point here is to note the difference between a polynomial and a polynomial function. In calculus, you often just think of these as the same thing, but in abstract algebra we really have to differentiate between the symbolic expression that is a polynomial versus the function it induces.)