Assignment 5 – Part 1 – Math 412

Textbook exercises:[1]

Section 3.3: 8

Other exercises:

(1) (WI) Let $f : R \to S$ be a ring homomorphism.

(a) Show that for all $r \in R$, $f(-r) = -f(r)$.

(b) Show that if $r \in R$ is a unit, then $f(r^{-1}) = f(r)^{-1}$.

(2) Let R and S be rings and suppose that $f : R \to S$ is a surjection such that for all $a, b \in R$,

$$f(a+b) = f(a) + f(b) \quad \text{and} \quad f(a \cdot b) = f(a) \cdot f(b).$$

Show that $f(1) = 1$. (Hint: the trick here is that the identity in S is characterized uniquely by its defining property, so show that $f(1)$ satisfies the property defining the identity in S.)

(3) Let X be a set. There is a standard bijection between the power set $\mathcal{P}(X)$ and the set of functions $X \to \{0, 1\}$, where the subset $Y \subseteq X$ corresponds to its so-called *characteristic function* χ_Y, namely the function

$$\chi_Y(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y. \end{cases}$$

Let's think of $\{0, 1\}$ as the ring with two elements, for instance think of $\{0, 1\} = \mathbf{Z}/2\mathbf{Z}$. Then, the set of functions $X \to \{0, 1\}$ is the *ring* of functions $\mathcal{F} := \text{Func}(\text{X}, \mathbf{Z}/2\mathbf{Z})$. Now, recall that we also have a ring structure on $\mathcal{P}(X)$ where addition is symmetric difference and multiplication is intersection. Show that the map $f : \mathcal{P}(X) \to \mathcal{F}$ sending Y to χ_Y is in fact a ring isomorphism.

(4) Let $\varphi : R \to S$ and $\psi : S \to T$ be two ring homomophisms. Show that $\psi \circ \varphi : R \to T$ is a ring homomorphism.

(5) (WI) Let $m, n \in \mathbf{Z}_{\geq 1}$ and suppose $n \mid m$. Define the function

$$\begin{array}{rcl} \varphi : \mathbf{Z}/m\mathbf{Z} & \to & \mathbf{Z}/n\mathbf{Z} \\ [a]_m & \mapsto & [a]_n. \end{array}$$

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*

(where $[a]_m$ denotes the congruence class of a modulo m). Show that this function is well-defined and a ring homomorphism.