Assignment 8 – All 2 parts – Math 412

Due in class: Thursday, Nov. 7, 2019

Textbook exercises:[1]

Section 6.1: 9, 10, 20(WI)

Section 6.2: 10, 18

Other exercises:

(1) (WI) Let p be a prime and let $I = (p, x)$ be the ideal generated by p and x in $R = \mathbf{Z}[x]$. Show that $R/I \cong \mathbf{Z}/p\mathbf{Z}$.

(2) Let p be a prime number and let $R = \mathbf{Z}_{(p)}$ be the ring from Assignment 4, Question 1 (and Assignment 6, Question 3(c)). Let $I = p\mathbf{Z}_{(p)}$ be the principal ideal generated by p in R. Show that $R/I \cong \mathbf{Z}/p\mathbf{Z}$. (Hint: you can start by showing that for $a/b \in \mathbf{Z}_{(p)}$, $\varphi(a/b) := [a] \cdot [b]^{-1} \in \mathbf{Z}/p\mathbf{Z}$ make sense and gives a ring homomorphism.)

(3) (WI) Let R be a commutative ring.

   (a) Show that the zero ideal is prime if and only if R is an integral domain.

   (b) Show that the zero ideal is maximal if and only if R is a field.

(4) Let p be a prime number and let $R = \mathbf{Z}_{(p)}$ be the ring from the previous question.

   (a) Show that $I = p\mathbf{Z}_{(p)}$ is a maximal ideal.

   (b) Show that it is the unique maximal ideal of R. (Hint: note that Questions 3(b) and 3(c) of Assignment 6 tell you that if $r \in \mathbf{Z}_{(p)}$ and $r \notin p\mathbf{Z}_{(p)}$, then r is a unit.) (Remark: a commutative ring with a unique maximal ideal is called a *local ring*.)

(5) Let F be a field and let $R = F[x]$. Recall that every ideal I of R is principal, i.e. of the form $I = (f(x))$ for some $f(x) \in F[x]$.

   (a) Let $I = (f(x))$ and $J = (g(x))$. Show that $I \supseteq J$ if and only if $f|g$.

   (b) Let $f(x) = x^2$ and let $\overline{R} = F[x]/(f(x))$. Show that $\overline{R}$ is a local ring with maximal ideal $([x])$. In fact, show that $([x])$ is the unique prime ideal of $\overline{R}$.

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*