Sample solutions in 'Mathematical English'

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Ex. 1.1.5.

Suppose

$$a = bq + r \quad \text{with } 0 \leq r < b.$$

Multiplying through by c yields

$$ac = bqc + rc.$$

We may rewrite this as

$$ac = (bc)q + (rc). \tag{1}$$

Since c is positive, multiplying the inequalities $0 \leq r < b$ by c gives

$$0 \leq rc < bc.$$

Then, by the Division Algorithm, Equation (1) says that q and rc are the (unique) quotient and remainder when ac is divided by bc. □

Ex. 1.1.10.

($\Rightarrow$): Suppose first that a and c have the same remainder when divided by n, i.e. there are nonnegative integers q_1, q_2, and r with $r < n$ such that

$$a = nq_1 + r \quad \text{and} \quad c = nq_2 + r.$$

Subtracting the second equation from the first yields

$$\begin{aligned} a - c &= n(q_1 - q_2) + (r - r) \\ &= n(q_1 - q_2) + 0. \end{aligned}$$

Taking $k = q_1 - q_2$ gives the desired result.
($\Leftarrow$): We argue via the contrapositive. So, suppose a and c have *different* remainders upon division by n, i.e. there are nonnegative integers q_i and r_i, with $i = 1$ or 2, $r_i < n$, and $r_1 \neq r_2$ such that

$$a = nq_1 + r_1 \quad \text{and} \quad c = nq_2 + r_2.$$

Subtracting the latter from the former gives

$$a - c = n(q_1 - q_2) + (r_1 - r_2).$$

Since the r_i are strictly less than n and distinct,

$$0 < |r_1 - r_2| < n.$$

We then have two cases: (i) $r_1 - r_2$ is positive, and (ii) $r_1 - r_2$ is negative.

Case (i): if $r_1 - r_2$ is positive, then $q := q_1 - q_2$ and $r := r_1 - r_2$ satisfy

$$a - c = nq + r, \quad 0 < r < n$$

so that, by the Division Algorithm, r is the remainder upon dividing $a - c$ by n. Since $r \neq 0$ is unique, there is no integer k such that $a - c = nk$.

Case (ii): if $r_1 - r_2$ is negative, we just need to shift by n, as follows. Let $q := q_1 - q_2 - 1$ and $r := r_1 - r_2 + n$. Then,

$$nq + r = n(q_1 - q_2) - n + r_1 + r_2 + n = a - c.$$

Since $-n < r_1 - r_2 < 0$, we have $0 < r < n$, so that again, by the Division Algorithm, r is the remainder upon dividing $a - c$ by n. Since $r \neq 0$ is unique, there is again no integer k such that $a - c = nk$. And we're done! $\square$