Assignment 10 – Part 1 – Math 612

Throughout, assume all rings are commutative.

(1) Suppose M is a free R-module

$$M = \bigoplus_{i \in I} Rm_i$$

and A is an R-algebra. Show that $A \otimes_R M$ is a free A-module with basis $1 \otimes m_i, i \in I$.

(2) Recall from Proposition 1.5 of the notes on tensor products that if A is an R-algebra and M is an R-module and N is an A-module, then $N \otimes_R M$ is an A-module via

$$a \cdot (n \otimes m) := (a \cdot n) \otimes m.$$

(a) Now, suppose further that P is an A-module. Show that

$$P \otimes_A (N \otimes_R M) \cong (P \otimes_A N) \otimes_R M$$

as A-modules.

(b) Conclude that if M is a flat R-module, then $A \otimes_R M$ is a flat A-module.

(c) Suppose that B is an A-algebra, A is an R-algebra, and M is an R-module. Show that

$$B \otimes_A (A \otimes_R M) \cong B \otimes_R M$$

as B-modules.

(3) Recall that an R-module N is called flat if the functor $- \otimes_R N$ is exact. N is called *faithfully flat* if this functor is also faithful. Show that the following are equivalent:

(i) N is faithfully flat,

(ii) N is flat and if $f : M_1 \to M_2$ is a non-zero homomorphism of R-modules, then the map $f \otimes \mathrm{id}_N : M_1 \otimes_R N \to M_2 \otimes_R N$ sending $m \otimes n$ to $f(m) \otimes n$ is non-zero,

(iii) N is flat and for all maximal ideals $\mathfrak{m}$ of R, we have that $M/\mathfrak{m}M \neq 0$,

(iv) a sequence $M_1 \to M_2 \to M_3$ is exact if and only if

$$M_1 \otimes_R N \to M_2 \otimes_R N \to M_3 \otimes_R N$$

is exact.