Assignment 11 – All 2 parts – Math 612

Due in class: Thursday, Apr. 28, 2016

Throughout, assume all rings are commutative.

(1) Let R be a commutative ring. Recall that the polynomial ring $R[x, y]$ satisfies a universal property stemming from the fact that it is the free commutative R-algebra on two elements, i.e. for every R-algebra A, there is a natural bijection

$$\mathrm{Hom}_{R\text{-alg}}(R[x, y], A) \cong \mathrm{Hom}_{\mathbf{Set}}(\{x, y\}, A).$$

In class, we stated that $R[x] \otimes_R R[y] \cong R[x, y]$. Prove this by showing that $R[x] \otimes_R R[y]$ satisfies the same universal property as $R[x, y]$.

(2) Determine the following tensor products.

(a) $\mathbf{Q}(i) \otimes_{\mathbf{Q}} \mathbf{R}$

(b) $\mathbf{Q}(i) \otimes_{\mathbf{Q}} \mathbf{C}$

(c) $\mathbf{Q}(i) \otimes_{\mathbf{Q}} \mathbf{Q}(i)$

(d) $\mathbf{Q}(i) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{2})$

(e) $\mathbf{Z}[i] \otimes_{\mathbf{Z}} \mathbf{F}_2$

(f) $\mathbf{Z}[i] \otimes_{\mathbf{Z}} \mathbf{F}_3$

(g) $\mathbf{Z}[i] \otimes_{\mathbf{Z}} \mathbf{F}_5$

(3) Let G be a group and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G over a field F. Suppose $\chi : G \to F^\times$ is a one-dimensional F-representation of G. Let

$$V^\chi := \{v \in V : \text{ for all } g \in G,\ g \cdot v = \chi(g)v\}.$$

Show that V^χ is a subrepresentation of V. It is called the *χ-isotypic component of V*. When χ is the trivial character, one usually writes V^G instead of V^χ.

(4) Let V be an F-representation of G, let W be a subrepresentation, and let V/W be the quotient vector space. Show that $g \cdot (v + W) := g \cdot v + W$ gives a well-defined action and turns V/W into a representation of G. It's called the *quotient representation*.

(5) Let S_n act on $X = \{1, 2, \dots, n\}$ in the usual way and let V_X be the associated linear representation over $\mathbf{C}$, i.e. the n-dimensional representation with basis $e_1, \dots, e_n$ where $\sigma \cdot e_i = e_{\sigma(i)}$.

(a) Let $U \subseteq V$ be the span of $e_1 + e_2 + \cdots + e_n$. Show that U is a subrepresentation of V and is isomorphic to the trivial representation.

(b) Let

$$W := \left\{ \sum_{i=1}^{n} a_i e_i \in V_X : \sum_{i=1}^{n} a_i = 0 \right\}.$$

Show that W is an irreducible subrepresentation of V. This is called the *standard representation of* S_n. (Hint: First, note that $\{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n\}$ is a basis of W. Then, take any non-zero vector w in W and, for each i, show that that are two permutations $\sigma_i, \tau_i \in S_n$ such that $\sigma_i(w) - \tau_i(w)$ is a non-zero multiple of $e_i - e_{i+1}$.)