

Assignment 12 – Part 1 – Math 612

- (1) (a) Let V be an F -representation of G . Suppose $V = W \oplus W'$ as representations. Show that $W \cong V/W'$.
- (b) Let $G = S_n$ and let V_X be the linear representation associated to the action of G on $X = \{1, 2, \dots, n\}$. Recall that

$$W := \left\{ \sum_{i=1}^n a_i e_i : \sum_{i=1}^n a_i = 0 \right\}$$

and

$$W' := \text{Span} \left(\sum_{i=1}^n e_i \right)$$

are subrepresentations of V_X , the former we called the standard representation of S_n . We also defined the standard representation by V_X/W' . Show that these two definitions give isomorphic representations.

- (2) (a) Let G be a group acting on a finite set X , let (ρ_X, V_X) be the associated linear representation over \mathbf{C} , and let χ be its character. For $g \in G$, show that $\chi(g)$ equals the number of fixed points of g acting on X .
- (b) Determine the character of the standard representation of S_3 . (Hint: here's an easy way: we'll see in class next time that if $V = W \oplus W'$, then $\chi_V = \chi_W + \chi_{W'}$. So, combining part (a) with what you probably did in Question (1b) allows you to determine the character of the standard representation.)
- (c) Determine the character of the standard representation of D_4 . Here, by the standard representation, I mean the 2-dimensional representation obtained by writing the elements of D_4 as rotation and reflection matrices.
- (3) Groups rings. Let G be a group and let R be a commutative ring. The free R -module on G can be turned into a (non-commutative if G is not abelian) ring $R[G]$, called the group ring of G over R , via

$$\left(\sum_{g \in G} a_g g \right) \cdot \left(\sum_{h \in G} b_h h \right) = \sum_{g \in G} \sum_{h \in G} a_g b_h gh,$$

i.e. like multiplying polynomials but where the variables multiply like the group operation.

- (a) Show that the function $G \rightarrow R[G]$ sending g to $1 \cdot g$ is injective and multiplicative.
- (b) Here's another way of defining the group ring. Let $C_c(G, R)$ be the set of functions $f : G \rightarrow R$ whose support is finite (the *support* of a function f is the set of $g \in G$ such that $f(g) \neq 0$). This is an R -module via

$$(r \cdot f_1 + f_2)(g) := r f_1(g) + f_2(g).$$

Define the *convolution product* of $f_1, f_2 \in C_c(G, R)$ by

$$(f_1 * f_2)(g_0) = \sum_{g \in G} f_1(g) f_2(g^{-1} g_0).$$

This makes $C_c(G, R)$ into a ring. Show that the map $C_c(G, R) \rightarrow R[G]$

$$f \mapsto \sum_{g \in G} f(g)g$$

is a ring isomorphism.

- (c) Now take $R = F$ a field. Let $g \in G$ act on $f \in C_c(G, R)$ by

$$(g \cdot f)(g_0) := f(g^{-1} g_0).$$

Show that this gives an F -representation of G ; in particular, show that this is a left action and that it is F -linear. (Equivalently, g acts on an element of $F[G]$ via

$$g \cdot \sum_{h \in G} a_h h := \sum_{h \in G} a_h g h,$$

i.e. by multiplication viewing g itself as the element $1 \cdot g \in F[G]$.) Show that this representation is isomorphic to the regular representation of G .

- (d) Let (ρ, V) be an F -representation of G . Show that V can be considered as an $F[G]$ -module via

$$\left(\sum_{g \in G} a_g g \right) \cdot v := \sum_{g \in G} a_g (g \cdot v).$$

Conversely, let M be an $F[G]$ -module. Show that M is an F -representation of G via

$$g \cdot m := (1 \cdot g) \cdot m.$$

- (e) Remarks: This sets up a very nice correspondence between the category of

$F[G]$ -modules and the category of F -representations of G . Morphisms correspond to morphisms. Submodules correspond to subrepresentation, direct sums correspond to direct sums, etc. And so, one also gets tensor products and everything else.