Assignment 12 – All 2 parts – Math 612

Due: Thursday, May 5, 2016, whatever that means

(1) (a) Let V be an F-representation of G. Suppose $V = W \oplus W'$ as representations. Show that $W \cong V/W'$.

(b) Let $G = S_n$ and let V_X be the linear representation associated to the action of G on $X = \{1, 2, \ldots, n\}$. Recall that

$$W := \left\{ \sum_{i=1}^{n} a_i e_i : \sum_{i=1}^{n} a_i = 0 \right\}$$

and

$$W' := \text{Span}\left(\sum_{i=1}^{n} e_i \right)$$

are subrepresentations of V_X, the former we called the standard representation of S_n. We also defined the standard representation by V_X/W'. Show that these two definitions give isomorphic representations.

(2) (a) Let G be a group acting on a finite set X, let (ρ_X, V_X) be the associated linear representation over $\mathbf{C}$, and let χ be its character. For $g \in G$, show that $\chi(g)$ equals the number of fixed points of g acting on X.

(b) Determine the character of the standard representation of S_3. (Hint: here's an easy way: we'll see in class next time that if $V = W \oplus W'$, then $\chi_V = \chi_W + \chi_{W'}$. So, combining part (a) with what you probably did in Question (1b) allows you to determine the character of the standard representation.)

(c) Determine the character of the standard representation of D_4. Here, by the standard representation, I mean the 2-dimensional representation obtained by writing the elements of D_4 as rotation and reflection matrices.

(3) Groups rings. Let G be a group and let R be a commutative ring. The free R-module on G can be turned into a (non-commutative if G is not abelian) ring $R[G]$, called the group ring of G over R, via

$$\left(\sum_{g \in G} a_g g \right) \cdot \left(\sum_{h \in G} b_h h \right) = \sum_{g \in G} \sum_{h \in G} a_g b_h gh,$$

i.e. like multiplying polynomials but where the variables multiply like the group operation.

(a) Show that the function $G \to R[G]$ sending g to $1 \cdot g$ is injective and multiplicative.

(b) Here's another way of defining the group ring. Let $C_c(G, R)$ be the set of functions $f : G \to R$ whose support is finite (the *support* of a function f is the set of $g \in G$ such that $f(g) \neq 0$). This is an R-module via

$$(r \cdot f_1 + f_2)(g) := r f_1(g) + f_2(g).$$

Define the *convolution product* of $f_1, f_2 \in C_c(G, R)$ by

$$(f_1 * f_2)(g_0) = \sum_{g \in G} f_1(g) f_2(g^{-1} g_0).$$

This makes $C_c(G, R)$ into a ring. Show that the map $C_c(G, R) \to R[G]$

$$f \longmapsto \sum_{g \in G} f(g) g$$

is a ring isomorphism.

(c) Now take $R = F$ a field. Let $g \in G$ act on $f \in C_c(G, R)$ by

$$(g \cdot f)(g_0) := f(g^{-1} g_0).$$

Show that this gives an F-representation of G; in particular, show that this is a left action and that it is F-linear. (Equivalently, g acts on an element of $F[G]$ via

$$g \cdot \sum_{h \in G} a_h h := \sum_{h \in G} a_h g h,$$

i.e. by multiplication viewing g itself as the element $1 \cdot g \in F[G]$.) Show that this representation is isomorphic to the regular representation of G.

(d) Let (ρ, V) be an F-representation of G. Show that V can be considered as an $F[G]$-module via

$$\left(\sum_{g \in G} a_g g \right) \cdot v := \sum_{g \in G} a_g (g \cdot v).$$

Conversely, let M be an $F[G]$-module. Show that M is an F-representation of G via

$$g \cdot m := (1 \cdot g) \cdot m.$$

(e) Remarks: This sets up a very nice correspondence between the category of

$F[G]$-modules and the category of F-representations of G. Morphisms correspond to morphisms. Submodules correspond to subrepresentation, direct sums correspond to direct sums, etc. And so, one also gets tensor products and everything else.

(4) (a) Write out the character table of C_3.

(b) Write out the character table of S_3. Let V be the standard representation of S_3. Decompose the representation $V \otimes V$ into irreducible representations.

(5) If G is a group, N is a normal subgroup, and (ρ, V) is a representation of G/N, note that ρ gives a representation of $\rho_G : G \to G/N \to \mathrm{GL}(V)$ by precomposing ρ with the reduction mod N map.

(a) Show that if ρ is irreducible, then so is ρ_G.

(b) Recall that $N := \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4)(1\ 4)(2\ 3)\}$ is a normal subgroup of S_4. Show that $S_4/N \cong S_3$. Let ρ be the standard representation of S_3. Show that ρ_{S_4} is the irreducible representation V_5 of S_4 whose character is the last row of the character table we wrote in class, i.e. the row

	e	$(1\ 2)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4)$	$(1\ 2)(3\ 4)$
V_5	2	0	-1	0	2

(6) The character table of A_4. Recall that the conjugacy classes of S_n correspond to the cycle types which in turn correspond to partitions of n. The conjugacy classes of A_n are subsets of the conjugacy classes of S_n and there is a nice criterion for how the conjugacy classes of S_n break up in A_n: if the conjugacy class of S_n corresponds to the partition $n = n_1 + n_2 + \cdots + n_k$, then it splits into two conjugacy classes of equal size in A_n if and only if the n_i are distinct odd integers; otherwise, the conjugacy remains unchanged in A_n.

(a) Determine the conjugacy classes of of A_4.

(b) Determine the character table of A_4. (Hint: you can get 3 distinct irreducible one-dimensional representations from taking $A_4/N \cong C_3$, where N is the subgroup in Question (5b), and precomposing with $A_4 \to A_4/N$.)