Assignment 2 – All 2 parts – Math 612

Due in class: Thursday, Jan. 28, 2016

(1) A D_4-extension. Let $K = \mathbf{Q}(\sqrt[4]{2})$ and let $\widetilde{K} = \mathbf{Q}(\sqrt[4]{2}, i)$ be its Galois closure (over $\mathbf{Q}$). For this exercise, I recommend you look back at everything we did with $\mathbf{Q}(\sqrt[3]{2})$ in the past two lectures. You'll also want to look back at problems (2) and (3) of Assignment 3 of last semester.

   (a) Show that $[\widetilde{K} : \mathbf{Q}] = 8$. Show that $\widetilde{K}/\mathbf{Q}$ is Galois and write down the 8 elements of the Galois group $G := \mathrm{Gal}(\widetilde{K}/\mathbf{Q})$.

   (b) Show that $G \cong D_4$. (Hint: for instance, you could find a couple generators and show that satisfy the correct relations.)

   (c) Write down the action of G on the four conjugates of $\sqrt[4]{2}$ (for instance, you could label the four roots $1, 2, 3, 4$ and write down an explicit isomorphism of G with a subgroup of S_4).

   (d) Work out the subgroup lattice of D_4 and the corresponding lattice of intermediate extensions of $\widetilde{K}/\mathbf{Q}$. In the latter, indicate which extensions are Galois and what their Galois groups are.

(2) A non-simple finite extension. Let p be a prime and let $F = \overline{\mathbf{F}}_p(u, v)$, where u and v are two variables and let $K = F(u^{1/p}, v^{1/p}) = \overline{\mathbf{F}}_p(u^{1/p}, v^{1/p})$.

   (a) Show that $[K : F] = p^2$.

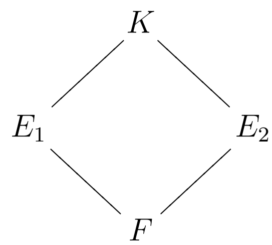
   (b) For $c \in \overline{F}_p$, let $E_c = F(u^{1/p} + cv^{1/p})$. Determine the minimal polynomial of $u^{1/p} + cv^{1/p}$ over F and conclude that $[E_c : F] = p$.

   (c) If there is $c_1 \neq c_2$ in $\overline{\mathbf{F}}_p$ such that $E_{c_1} = E_{c_2}$, then our proof of the Primitive Element Theorem shows that $K = F(u^{1/p} + c_1 v^{1/p})$. Explain why this is a contradiction.

   (d) Conclude that K/F is not a simple extension.

(3) Intersections and composita in Galois extensions. Suppose we have the following

diagram of field extensions:



where K/F is Galois with Galois group G and E_i corresponds to $H_i \leq G$ (i.e. $E_i = K^{H_i}$).

(a) Show that $E_1 \cap E_2$ corresponds to $\langle H_1, H_2 \rangle$ (the subgroup of G generated by H_1 and H_2).

(b) Show that $H_1 \cap H_2$ corresponds to $E_1 E_2$.

(4) Transitive subgroups. A *transitive subgroup* of S_n is a subgroup $H \leq S_n$ such that the action of H on $\{1, 2, \ldots, n\}$ is transitive. We'll see that Galois groups of separable irreducible polynomials of degree n are necessarily transitive subgroups of S_n (i.e. their action on the roots of the polynomial is transitive).

(a) Show that the only two transitive subgroups of S_3 are $C_3 = \langle (1\ 2\ 3) \rangle$ and S_3.

(b) For all $n \geq 3$, show that A_n is a transitive subgroup of S_n.

(c) Let $H := \langle (1\ 2)(3\ 4), (2\ 4) \rangle \leq S_4$. Show that $H \cong D_4$ and explain why H is a transitive subgroup of S_4.

(d) Do you remember the normal subgroup of A_4 isomorphic to V_4? Show that it's a transitive subgroup of S_4. But also show that there is a non-transitive subgroup of S_4 isomorphic to V_4.

(e) An isomorphism class of groups may be a transitive subgroup of S_n for more than one n. For instance, show that S_3 can also be viewed as a transitive subgroup of S_6.

(f) Show that every finite group is a transitive subgroup of S_n for some n.

(5) Abelian and cyclic extensions. A Galois extension K/F is called *abelian* (resp. *cyclic*) if $\mathrm{Gal}(K/F)$ is abelian (resp. cyclic). Show that any intermediate extension E/F of an abelian (resp. cyclic) extension is abelian (resp. cyclic).