Assignment 3 – Part 1 – Math 612

(1) Let F be a field and let $f(x) \in F[x]$ be an irreducible and separable polynomial of degree $n \geq 1$. Let $K = F(\alpha)$ where α is a root of f.

(a) Say $n = 4$ and the Galois group of f is D_4. Show that $\mathrm{Aut}(K/F) \cong C_2$.

(b) Suppose the Galois group of f is S_n. Show that $\mathrm{Aut}(K/F)$ is trivial.

(2) Prove the following claim we stated but didn't prove in class: if K/F is Galois with Galois group G and H is a not necessarily normal subgroup corresponding to the intermediate extension E, then there is a natural bijection between the set of F-embeddings of E into $\overline{K}$ and the set of cosets G/H.

(3) In this exercise, you will show that S_4 is a transitive subgroup of S_6 in two quite different ways, in particular, there are two conjugacy classes of transitive S_4 in S_6.

(a) First off, recall that S_4 is the rotational symmetry group of the cube (the four things it acts on are the pairs of opposite vertices). Show that S_4 acts faithfully and transitively on the 6 faces of a cube, thus making S_4 into a transitive subgroup of S_6. Show that the image in S_6 is not contained in A_6.

(b) By definition, S_4 acts on the set $X = \{1, 2, 3, 4\}$. Let X_2 be the set of subsets of X of cardinality 2. Show that S_4 acts faithfully and transitively on X_2, thus making it again a transitive subgroup of S_6. (A geometric approach as above can also be taken: S_4 is in fact the group of rotational *and reflectional* (is that a word?) symmetries of a tetrahedron and you can show that it acts faithfully and transitively on the 6 edges.)

(c) In latter case, show that the image lies in A_6. (Hint: it may be helpful for you to know that S_4 is generated by $(1\ 2\ 3\ 4)$ and $(1\ 2)$.)

(d) Conclude that these two copies of S_4 in S_6 are not conjugate. Since the second one is in A_6, its conjugacy class is usually denoted S_4^+, while the other is denoted S_4^-. (Hint: A_6 is normal.)

(4) S_5 in S_6. Show that S_5 is a transitive subgroup of S_6. (Hint: consider the action of S_5 on the set of its Sylow 5-subgroups.)