Assignment 5 – All 2 parts – Math 612

Due in class: Thursday, Feb. 18, 2016

(1) Let F be a field, K/F a finite Galois extension and E/F any other finite extension such that $E \cap K = F$ (and K and E are contained in a common field). Assume KE/F is Galois. Show that $\mathrm{Gal}(KE/F) \cong \mathrm{Gal}(KE/K) \rtimes \mathrm{Gal}(K/F)$.

(2) There's a group I somehow managed not to discuss last semester: the quaternion group Q_8. It's the other non-abelian group of order 8 (the other being D_4). I didn't mention it because it's not easy to define without having first talked about the quaternions $\mathbf{H}$. With $\mathbf{H}$, it's straightforward. Recall that the quaternions are

$$\mathbf{H} := \{a + bi + cj + dk : a, b, c, d \in \mathbf{R}\},$$

where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$ (so that for any permutation σ of $\{i, j, k\}$, one has $\sigma(i)\sigma(j) = \mathrm{sgn}(\sigma)\sigma(k)$). Then, the quaternion group is

$$Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}.$$

One can also give a presentation very similar to D_4:

$$Q_8 \cong \langle x, y \mid x^4 = 1, y^2 = x^2, y^{-1}xy = x^{-1}\rangle$$

(D_4 has $y^2 = 1$ instead). Here, you can take $x = i$ and $y = j$, for instance. Anyway.

(a) Show that any non-trivial subgroup of Q_8 contains $\{\pm 1\}$. Conclude that Q_8 is not a semi-direct product (of non-trivial subgroups; of course, $Q_8 = Q_8 \times 1$).

(b) Show that every subgroup of Q_8 is normal. Even though it's not abelian. Wild, innit?

(c) In the previous question, you showed $\mathrm{Gal}(KE/F)$ is a semi-direct product. But that precludes that $\mathrm{Gal}(KE/F) \cong Q_8$. So, if L/F is a Galois extension with group Q_8 and K/F is an intermediate extension that is Galois and E/F is some other intermediate extension such that $L = KE$, what is it that makes it so that Q_8 not being a semidirect product doesn't contradict the conclusion of question (1)?

(3) Generators of S_n. Let $n \in \mathbf{Z}_{\geq 2}$.

(a) Given a transposition $(i\ j) \in S_n$ and some $\sigma \in S_n$, show that

$$\sigma(i\ j)\sigma^{-1} = (\sigma(i)\ \sigma(j)).$$

(b) Show that S_n is generated by the transpositions

$$(1\ 2), (2\ 3), \ldots, (n-1\ n).$$

(Hint: show you can get any other transposition by conjugating.)

(c) Show that S_n is generated by the $(n-1)$-cycle $(1\ 2\ 3\ \cdots\ n-1)$ and any given transposition $(i\ n)$ for $i \neq n$.

(d) Conclude that a *transitive* subgroup of S_n that contains a transposition and an $(n-1)$-cycle is S_n.

(4) Determine the Galois groups of the following polynomials over $\mathbf{Q}$.

(a) $f(x) = x^5 - 30x^4 + 15x^2 + 5x + 30$.

(b) $f(x) = x^3 + 6x^2 - 12x + 3$.