**Assignment 6 – All 2 parts – Math 612**

**Due in class: Thursday, Feb. 25, 2016**

(1) Let $d \in \mathbf{Z}$ be squarefree and let $K_d = \mathbf{Q}(\sqrt{d})$. Let

$$\mathcal{O}_d = \begin{cases} \left\{a + b\sqrt{d} \in K_d : a, b \in \mathbf{Z}\right\} & \text{if } d \equiv 2, 3 \pmod 4 \\ \left\{a + b\left(\dfrac{1+\sqrt{d}}{2}\right) \in K_d : a, b \in \mathbf{Z}\right\} & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

For $\alpha \in K_d$, show that $\mathrm{Tr}_{K_d/\mathbf{Q}}(\alpha), N_{K_d/\mathbf{Q}}(\alpha) \in \mathbf{Z}$ if and only if $\alpha \in O_d$.

(2) Let $K = \mathbf{Q}(\sqrt[3]{2})$. What are $N_{K/\mathbf{Q}}(\sqrt[3]{2})$ and $\mathrm{Tr}_{K/\mathbf{Q}}(\sqrt[3]{2})$?

(3) Let $p$ be a prime. Show that all finite $p$-groups (that is, groups of order $p^n$ for some $n$) are solvable. (Hint: recall that the centre of a $p$-group is non-trivial.)

(4) Let $F$ be a field of characteristic 0. For a finite extension $K/F$, we will say that $K/F$ is *solvable by radicals* if $K$ is contained in a field $E$ such that $E/F$ is a root extension (recall from class that $E/F$ is a root extension if $E/F$ is finite and is at the top of a tower $E = E_m/E_{m-1}/\cdots/E_1/E_0 = F$ such that $E_i = E_{i-1}(\sqrt[n_i]{a_i})$ for some $a_i \in E_{i-1}$). Show that the class of finite extensions of $F$ that are solvable by radicals forms a distinguished class of extensions.

(5) Recall that in class we defined a finite group $G$ to be solvable if it has a filtration

$$G = G_0 \geq G_1 \geq \cdots \geq G_m = 1$$

such that $G_i/G_{i+1}$ is cyclic. Show that a finite group $G$ is solvable if and only if it has a filtration

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

such that $G_i/G_{i+1}$ is abelian.

(6) Given a group $G$ and two element $g, h \in G$, their *commutator* is $[g,h] := g^{-1}h^{-1}gh$. For two subsets $S, T \subseteq G$, define their *commutator subgroup* to be

$$[S,T] := \langle [s,t] : s \in S, t \in T\rangle \leq G.$$

Define the *commutator subgroup of* $G$ to be $G^{(1)} := [G,G]$ (also called the *derived subgroup* and denoted $G^{\mathrm{der}}$ or even $G'$).

(a) Show that $[G, G]$ is a normal subgroup of $G$ and that $G/[G, G]$ is abelian.

(b) Show that $G/[G, G]$ is the maximal abelian quotient of $G$, meaning that if $H \trianglelefteq G$ and $G/H$ is abelian, then $[G, G] \leq H$ (so that $G/[G, G]$ surjects onto $G/H$).

(c) The *derived series of* $G$ is the filtration $G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots$ where, for $i \geq 1$,

$$G^{(i)} := [G^{(i-1)}, G^{(i-1)}].$$

Show that if $G^{(n)} = 1$ for some $n \geq 0$, then $G$ is solvable.

(d) Conversely, show that if $G$ is solvable, then $G^{(n)} = 1$ for some $n$. (Hint: if $G = G_0 \geq G_1 \geq \cdots \geq G_m = 1$ is a filtration such that $G_i/G_{i+1}$ is abelian, show that $G^{(i)} \leq G_i$.)

(e) This is just a remark, I won't make you prove these things, though they are not difficult. Suppose

$$1 \to N \to G \to H \to 1$$

is a short exact sequence of groups. Then,

$G$ is solvable if and only if both $H$ and $N$ are solvable.

In fact, one can show that $G^{(i)}$ mod $N = (G/N)^{(i)}$ and, for any subgroup $N$ of $G$, normal or not, $N^{(i)} \leq G^{(i)}$.