Assignment 7 – All 2 parts – Math 612

Due in class: Thursday, Mar. 3, 2016

(1) Let $d \in \mathbf{Z}$ be squarefree and let $K_d = \mathbf{Q}(\sqrt{d})$. Find a normal basis of $K_d/\mathbf{Q}$.

(2) The Fundamental Theorem of Algebra: $\mathbf{C}$ is algebraically closed. This exercise develops an algebraic proof of the fundamental theorem of algebra. By necessity, an analytic input is required, so you may assume the following simple consequence of the Intermediate Value Theorem: if $f(x) \in \mathbf{R}[x]$ has odd degree, then it has a root in $\mathbf{R}$.

(a) Let σ denote complex conjugation. Let $f(x) \in \mathbf{C}[x]$. Show that $f(x)$ has a root in $\mathbf{C}$ if and only if $f^\sigma(x)$ has a root in $\mathbf{C}$. Conclude that to prove that $\mathbf{C}$ is algebraically closed, it suffices to show that every $f(x) \in \mathbf{R}[x]$ has a root in $\mathbf{C}$.

(b) Let $f(x) \in \mathbf{R}[x]$ and let K be its splitting field. Show that $K(i)/\mathbf{R}$ is Galois and that its Galois group is a 2-group. Conclude that $\mathrm{Gal}(K(i)/\mathbf{C})$ is a 2-group. (Hint: you can show that $\mathbf{R}$ has no proper extensions of odd degree.)

(c) Show that if $\mathrm{Gal}(K(i)/\mathbf{C}) \neq 1$, then there must be an intermediate extension $K(i)/E/\mathbf{C}$ with $E/\mathbf{C}$ of degree 2.

(d) Show that $\mathbf{C}$ has no degree 2 extensions. Conclude that $\mathbf{C}$ is algebraically closed. (Hint: the quadratic formula!)

(3) Let B/A be an integral ring extension.

(a) Let $a \in A$. Suppose a is a unit in B, show that a is in fact a unit in A.

(b) Suppose $f : B \to C$ is a ring homomorphism. Show that $f(B)$ is integral over $f(A)$.

(c) If I is an ideal of B and $I^c := I \cap A$ is its contraction, show that B/I is integral over A/I^c (to begin with, explain why B/I is an algebra over A/I^c).

(4) Let A be an integral domain, let F be its field of fractions, let K/F be an algebraic extension of fields, and let $\alpha \in K$. Show that α is integral over A if and only if the coefficients of its minimal polynomial over F are integral over A.