**Assignment 8 – Part 1 – Math 612**

(1) As in class, let $F$ be a field, let $F[C] = F[x,y]/(y^2 - x^3)$ (in class, we had $C$ be the curve given by $y^2 - x^3$), let $F(C)$ be the field of fractions of $F[C]$, and let $\overline{x}$ and $\overline{y}$ be the images of $x, y \in F[x,y]$ in $F[C]$. In class, we showed that $\overline{z} := \overline{y}/\overline{x} \in F(C)$ is integral over $F[C]$ and claimed that it is not in $F[C]$. Prove the latter statement.

(2) Let $A$ be an integral domain and let $F$ be its field of fractions. Suppose $K/F$ is a field extension and $\alpha \in K$ is algebraic over $F$. Show that there is a non-zero element $a \in A$ such that $a\alpha$ is integral over $A$.

(3) Here are a few things we'll need soon. Let $A$ be a (commutative) ring.

(a) Let $I$ be an ideal in $A$. The *radical of* $I$ is

$$\operatorname{rad}(I) := \{a \in A : a^n \in I \text{ for some } n \in \mathbf{Z}_{\geq 1}\}.$$

Show that the radical of an ideal is an ideal.

(b) The *nilradical of* $A$ is the radical of the zero ideal, denoted $\operatorname{nil}(A)$. It is the set of nilpotent elements of $A$. A ring is called *reduced* if it has no non-zero nilpotent elements. Show that $A/\operatorname{nil}(A)$ is reduced.

(c) An ideal is called a *radical ideal* if $I = \operatorname{rad}(I)$. Show that $\operatorname{rad}(I)/I$ is the nilradical of $A/I$.

(d) Show that the ideal $(n)$ of $\mathbf{Z}$ is radical if and only if $n$ is squarefree.