Assignment 8 – All 2 parts – Math 612

Due in class: Thursday, Mar. 10, 2016

(1) As in class, let F be a field, let $F[C] = F[x,y]/(y^2 - x^3)$ (in class, we had C be the curve given by $y^2 - x^3$), let $F(C)$ be the field of fractions of $F[C]$, and let $\overline{x}$ and $\overline{y}$ be the images of $x, y \in F[x,y]$ in $F[C]$. In class, we showed that $\overline{z} := \overline{y}/\overline{x} \in F(C)$ is integral over $F[C]$ and claimed that it is not in $F[C]$. Prove the latter statement.

(2) Let A be an integral domain and let F be its field of fractions. Suppose K/F is a field extension and $\alpha \in K$ is algebraic over F. Show that there is a non-zero element $a \in A$ such that $a\alpha$ is integral over A.

(3) Here are a few things we'll need soon. Let A be a (commutative) ring.

(a) Let I be an ideal in A. The *radical of* I is

$$\operatorname{rad}(I) := \{a \in A : a^n \in I \text{ for some } n \in \mathbf{Z}_{\geq 1}\}.$$

Show that the radical of an ideal is an ideal.

(b) The *nilradical of* A is the radical of the zero ideal, denoted $\operatorname{nil}(A)$. It is the set of nilpotent elements of A. A ring is called *reduced* if it has no non-zero nilpotent elements. Show that $A/\operatorname{nil}(A)$ is reduced.

(c) An ideal is called a *radical ideal* if $I = \operatorname{rad}(I)$. Show that $\operatorname{rad}(I)/I$ is the nilradical of A/I.

(d) Show that the ideal (n) of $\mathbf{Z}$ is radical if and only if n is squarefree.

(4) Let F be a field (not necessarily algebraically closed; nothing we've said in class so far required F to be algebraically closed). Recall that for $S \subseteq F[x_1, \ldots, x_n]$

$$\mathcal{V}(S) = \{a \in \mathbf{A}^n(F) : f(a) = 0 \text{ for all } f \in S\}$$

and for V any subset of $\mathbf{A}^n(F)$,

$$\mathcal{I}(V) = \{f \in F[x_1, \ldots, x_n] : f(v) = 0 \text{ for all } v \in V\}.$$

(a) Show that for subsets V, V_1, V_2 of $\mathbf{A}^n(F)$ and I an ideal of $F[x_1, \ldots, x_n]$

(i) If $V_1 \subseteq V_2$, then $\mathcal{I}(V_1) \supseteq \mathcal{I}(V_2)$.

(ii) $\mathcal{I}(V_1 \cup V_2) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2)$.

(iii) $V \subseteq \mathcal{V}(\mathcal{I}(V))$ and $I \subseteq \mathcal{I}(\mathcal{V}(I))$.

(iv) $\mathcal{V}(\mathcal{I}(\mathcal{V}(I))) = \mathcal{V}(I)$ and $\mathcal{I}(\mathcal{V}(\mathcal{I}(V))) = \mathcal{I}(V)$, i.e. if V is an algebraic set, then $\mathcal{V}(\mathcal{I}(V)) = V$, and if I is the ideal of some set V, then $\mathcal{I}(\mathcal{V}(I)) = I$.

(b) In class, in proving that an algebraic set is irreducible if and only if its ideal is prime, I used the following fact: If $V \subsetneq W$ are two algebraic sets, then $\mathcal{I}(V) \supsetneq \mathcal{I}(W)$. Prove this. (Hint: (iv) above.)

(c) Show that for any ideal I in $F[x_1, \ldots, x_n]$, $\mathcal{V}(I) = \mathcal{V}(\mathrm{rad}(I))$. Show that for any subset V of $\mathbf{A}^n(F)$, $\mathcal{I}(V)$ is a radical ideal.

(d) Show that for any field F, the algebraic sets in $\mathbf{A}^1(F)$ are $\emptyset, \mathbf{A}^1(F)$, and the finite subsets of $\mathbf{A}^1(F)$.

(e) Show that prime ideals are radical. Show that (x^2+1) is a radical ideal in $\mathbf{R}[x]$, but (x^2+1) is not of the form $\mathcal{I}(V)$ for any algebraic set in $\mathbf{A}^1(\mathbf{R})$.

(5) In Question 10 of Assignment 6 of last semester, you showed some basic facts about ideals with respect to the natural map $\iota : A \to S^{-1}A$, where A is a (commutative) ring and $S \subseteq A$ is a multiplicative subset. For I an ideal of A, let I^e be its extension to $S^{-1}A$, i.e. the ideal of $S^{-1}A$ generated by $\iota(I)$, and for an ideal J of $S^{-1}A$, let J^c be its contraction to A, i.e. $J^c = \iota^{-1}(J)$. You showed that $I^e = S^{-1}I = \{a/s : a \in I, s \in S\}$ and $J = (J^c)^e$. You also showed that $I^e = S^{-1}A$ if and only if $I \cap S \neq \emptyset$. Conclude from these facts that extension and contraction give inclusion-preserving bijections (that are inverses of each other) between the set of primes ideals of $S^{-1}A$ and the set of primes ideals of A that don't intersect S. Then, conclude that if $\mathfrak{p}$ is a prime ideal of A, then the prime ideals of $A_{\mathfrak{p}}$ are in order-preserving bijection with those of A contained in $\mathfrak{p}$.