Assignment 9 – Part 1 – Math 612

(1) Let R be a commutative ring and let M and N be two R-modules.

(a) Let $m \otimes n \in M \otimes_R N$. Show that $m \otimes n = 0$ if and only if for all R-modules X and all R-bilinear forms $\psi : M \times N \to X$, we have $\psi(m, n) = 0$. (Hint: recall that the natural map $\psi_{\text{univ}} : M \times N \to M \otimes_R N$ sending (m, n) to $m \otimes n$ is itself an R-bilinear map.)

(b) Show that $M \otimes_R N = 0$ if and only if for all R-modules X and all R-bilinear forms $\psi : M \times N \to X$, we have $\psi(m, n) = 0$ for all $(m, n) \in M \times N$.

(2) In this exercise, you'll prove the following property of tensor products stated in class: let R be a commutative ring, let I be an ideal in R, and let M be an R-module, then

$$R/I \otimes_R M \cong M/IM.$$

(a) First, show that every element of $R/I \otimes_R M$ is a pure tensor of the form $1 \otimes m$.

(b) Now, show that $R/I \otimes_R M \cong M/IM$. (Hint: to get the map from left to right, consider the map $R/I \times M \to M/IM$ sending $(r + I, m)$ to $rm + IM$). Part (a) is helpful in getting a map from right to left).

(3) Again, let R be a commutative ring.

(a) Suppose I is an ideal of R that contains a non-zero element that is not a zero divisor. Show that R/I is not flat.

(b) Suppose M is a flat R-module. Show that M is torsion-free (recall that torsion-free means that for all $m \neq 0$ in M and all $r \neq 0$ in R, $rm = 0$ implies r is a zero-divisor).

(4) In the Snake lemma, we have the commutative diagram

$$\begin{array}{ccccccc} & & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_3} & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & & \end{array}$$

whose rows are exact and obtained an exact sequence

$$\ker d_1 \longrightarrow \ker d_2 \longrightarrow \ker d_3 \xrightarrow{\delta} \operatorname{coker} d_1 \longrightarrow \operatorname{coker} d_2 \longrightarrow \operatorname{coker} d_3.$$

(a) In class, we showed the exactness at $\ker d_3$. Show exactness at $\ker d_2, \operatorname{coker} d_1$, and $\operatorname{coker} d_2$.

(b) Show that if $M_1 \to M_2$ is injective, then so is $\ker d_1 \to \ker d_2$. Show that if $N_2 \to N_3$ is surjective, then so is $\operatorname{coker} d_2 \to \operatorname{coker} d_3$.