Assignment 9 – All 2 parts – Math 612

Due in class: Thursday, Apr. 14, 2016

(1) Let R be a commutative ring and let M and N be two R-modules.

(a) Let $m \otimes n \in M \otimes_R N$. Show that $m \otimes n = 0$ if and only if for all R-modules X and all R-bilinear forms $\psi : M \times N \to X$, we have $\psi(m, n) = 0$. (Hint: recall that the natural map $\psi_{\text{univ}} : M \times N \to M \otimes_R N$ sending (m, n) to $m \otimes n$ is itself an R-bilinear map.)

(b) Show that $M \otimes_R N = 0$ if and only if for all R-modules X and all R-bilinear forms $\psi : M \times N \to X$, we have $\psi(m, n) = 0$ for all $(m, n) \in M \times N$.

(2) In this exercise, you'll prove the following property of tensor products stated in class: let R be a commutative ring, let I be an ideal in R, and let M be an R-module, then

$$R/I \otimes_R M \cong M/IM.$$

(a) First, show that every element of $R/I \otimes_R M$ is a pure tensor of the form $1 \otimes m$.

(b) Now, show that $R/I \otimes_R M \cong M/IM$. (Hint: to get the map from left to right, consider the map $R/I \times M \to M/IM$ sending $(r + I, m)$ to $rm + IM$). Part (a) is helpful in getting a map from right to left).

(3) Again, let R be a commutative ring.

(a) Suppose I is an ideal of R that contains a non-zero element that is not a zero divisor. Show that R/I is not flat.

(b) Suppose M is a flat R-module. Show that M is torsion-free (recall that torsion-free means that for all $m \neq 0$ in M and all $r \neq 0$ in R, $rm = 0$ implies r is a zero-divisor).

(4) In the Snake lemma, we have the commutative diagram

$$\begin{array}{ccccccc} & & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_3} & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & & \end{array}$$

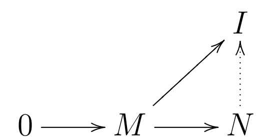
whose rows are exact and obtained an exact sequence

$$\ker d_1 \longrightarrow \ker d_2 \longrightarrow \ker d_3 \xrightarrow{\delta} \operatorname{coker} d_1 \longrightarrow \operatorname{coker} d_2 \longrightarrow \operatorname{coker} d_3.$$

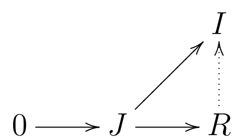
(a) In class, we showed the exactness at $\ker d_3$. Show exactness at $\ker d_2, \operatorname{coker} d_1$, and $\operatorname{coker} d_2$.

(b) Show that if $M_1 \to M_2$ is injective, then so is $\ker d_1 \to \ker d_2$. Show that if $N_2 \to N_3$ is surjective, then so is $\operatorname{coker} d_2 \to \operatorname{coker} d_3$.

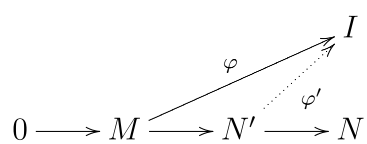
(5) Baer's criterion. Recall that an R-module I is called *injective* if given any injective homomorphism $M \to N$ of R-modules, any homomorphism $M \to I$ can be extended to a homomorphism $N \to I$ making the diagram



commute. Prove Baer's criterion: I is injective if and only if for every non-zero ideal J of R, any R-module homomorphism $J \to I$ can be extended to a homomorphism $R \to I$ making the diagram



(Hint: for the hard direction, you'll want to use Zorn's lemma. Specifically, given an injective homomorphism $M \to N$, which we'll just think of as $M \subseteq N$, and a homomorphism $\varphi : M \to I$, consider the partially ordered set X of pairs (N', φ') such that φ' extends φ part of the way to N, i.e. $M \subseteq N' \subseteq N$ and we have the commutative diagram



commute. We define $(N', \varphi') \leq (N'', \varphi'')$ if $N' \subseteq N''$ and $\varphi''|_{N'} = \varphi'$. You want to show a maximal element of X is the map $N \to I$ you want. You need to reduce this to the statement for ideals. If (N', φ') is a maximal element and there is an $n' \in N \setminus N'$, consider the ideal $J' := \{r \in R : rn' \in N'\}$.)

(6) An R-module D is called *divisible* if for every non-zero $r \in R$ that is not a zero-divisor and every $d \in D$, there is a $d' \in D$ such that $d = rd'$ (i.e. you can 'divide by r', though the answer need not be unique). Equivalently, D is divisible if for all r as above, the 'multiplication by r' map $m_r : D \to D$ sending d to rd is surjective.

(a) Show that an injective R-module I is divisible. (Hint: if $d \in I$ and $r \in R$ is a non-zero non-zerodivisor, consider the R-linear map $\varphi_d : Rr \to I$ defined by sending ar to ad.)

(b) Conversely, suppose R is a PID and show that if D is divisible, then it is injective. (Hint: use Baer's criterion, then you can do something similar to part (a).)

(c) Conclude that $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are injective $\mathbf{Z}$-modules, but, for all $n \in \mathbf{Z}_{\geq 2}$, $\mathbf{Z}/n\mathbf{Z}$ is not an injective $\mathbf{Z}$-module and neither is $\mathbf{Z}$ itself.

(7) $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$. In this exercise, you'll compute the $\mathbf{Z}$-module $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$ for $m, n \in \mathbf{Z}_{\geq 2}$. Recall that $\operatorname{Ext}^1_{\mathbf{Z}}(-, \mathbf{Z}/n\mathbf{Z})$ is the first right derived functor of $\operatorname{Hom}_{\mathbf{Z}}(-, \mathbf{Z}/n\mathbf{Z})$ and hence the Ext groups can be computed by taking an injective resolution of $\mathbf{Z}/n\mathbf{Z}$.

(a) Show that

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \mathbf{Q}/\mathbf{Z} & \longrightarrow & \mathbf{Q}/\mathbf{Z} & \longrightarrow & 0 \\
 & & 1 & \longmapsto & \frac{1}{n} + \mathbf{Z} & & & & \\
 & & & & a + \mathbf{Z} & \longmapsto & na + \mathbf{Z} & &
\end{array}
$$

is an injective resolution of $\mathbf{Z}/n\mathbf{Z}$.

(b) Show that $\operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \cong (\frac{1}{m}\mathbf{Z})/\mathbf{Z} \cong \mathbf{Z}/m\mathbf{Z}$.

(c) Applying $\operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, -)$ to the complex

$$0 \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow 0$$

you get the complex

$$0 \longrightarrow \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \xrightarrow{d^0} \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \xrightarrow{d^1} 0.$$

Recall that $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}) := \ker d^1 / \operatorname{im} d^0$. Show that

$$\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}) \cong (\mathbf{Z}/m\mathbf{Z})\big/(n(\mathbf{Z}/m\mathbf{Z})) \cong \mathbf{Z}/\gcd(m,n)\mathbf{Z}.$$

(d) This is just a remark. There is a nice interpretation of $\operatorname{Ext}^1_R(M, N)$. Given two R-modules M and N, an *extension of M by N* is an R-module E together with a short exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0.$$

Two extensions E and E' are called *equivalent* if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the double vertical lines are equals signs indicating that those maps are the identity maps (this diagram implies the middle map is an isomorphism). The set of equivalence classes of extensions is made into an abelian group using the so-called *Baer sum*. This group is isomorphic to $\operatorname{Ext}^1_R(M,N)$, hence the name of the latter. The isomorphism is given 'explicitly' as follows. Given the short exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0.$$

and applying $\operatorname{Hom}_R(M,-)$, the long exact sequence in cohomology includes the connecting homomorphism

$$\operatorname{Hom}_R(M,M) \xrightarrow{\delta} \operatorname{Ext}^1_R(M,N).$$

The isomorphism is obtained by sending the extension E to $\delta(\mathrm{id}_M)$.