Assignment 11 – All 2 parts – Math 612

Due in class: Thursday, Apr. 25, 2019

(1) In class, we considered $K = \mathbf{Q}(\sqrt{3}, \sqrt{5})$ and showed that for $\alpha = \sqrt{3} + \sqrt{5}$, we have $K = \mathbf{Q}(\alpha)$. We also showed that $\beta = \sqrt{3} - \sqrt{5}$ is another root of the minimal polynomial $x^4 - 16x^2 + 4$ of α. Since we also showed $K/\mathbf{Q}$ is Galois, it must be that β is a polynomial in α with coefficients in $\mathbf{Q}$. Find such a polynomial.

(2) In class, we considered $K = \mathbf{Q}(\sqrt[3]{2})$ and its Galois closure (over $\mathbf{Q}$) $K^{gal} = \mathbf{Q}(\sqrt[3]{2}, \omega)$, which is the splitting field of $f(x) = x^3 - 2$.

(a) Let $\alpha = \omega + \sqrt[3]{2}$. Show that $K = \mathbf{Q}(\alpha)$.

(b) Let $G = \mathrm{Gal}(f)$ which we think of as acting on the roots of $f(x)$, which we denote $r_j = \omega^j \sqrt[3]{2}$, for $j = 0, 1, 2$. In class, we found the six elements of G, which we denoted $\tau_{\pm,j}$ for $j = 0, 1, 2$. These elements were extensions of $\sigma_\pm : \mathbf{Q}(\omega) \to \mathbf{Q}(\omega)$, where $\sigma_\pm(\omega) = \omega^{\pm 1}$. Then,

$$\tau_{\pm,j}(\sqrt[3]{2}) = r_j.$$

We showed that $G \cong \mathrm{Aut}(\{r_0, r_1, r_3\}) \cong S_3$. Write out the explicit permutations that each of the $\tau_{\pm,j}$ corresponds to.

(3) A non-simple finite extension. Let p be a prime and let $F = \overline{\mathbf{F}}_p(u, v)$, where u and v are two variables and let $K = F(u^{1/p}, v^{1/p}) = \overline{\mathbf{F}}_p(u^{1/p}, v^{1/p})$.

(a) Show that $[K : F] = p^2$. (Hint: use a tower of extension and Eisenstein's criterion.)

(b) Let $\alpha \in K$ and show that $\alpha^p \in F$.

(c) Conclude that K/F is not a simple extension. (Hint: the previous part gives an upper bound on the degree of the minimal polynomial (over F) of any $\alpha \in K$.)