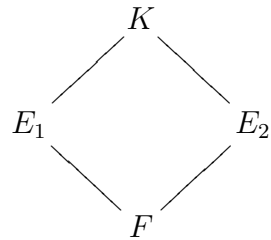


Assignment 12 – Part 1 – Math 612

- (1) A D_4 -extension. Let $K = \mathbf{Q}(\sqrt[4]{2})$ and let $\tilde{K} = \mathbf{Q}(\sqrt[4]{2}, i)$. For this exercise, I recommend you look back at everything we did in class with $\mathbf{Q}(\sqrt[3]{2})$. You'll probably also want to look back at problems (1) and (2) of Assignment 4 of last semester for material on dihedral groups.
- (a) Show that $[\tilde{K} : \mathbf{Q}] = 8$. Show that \tilde{K}/\mathbf{Q} is Galois (and hence the Galois closure of K/\mathbf{Q}) and write down the 8 elements of the Galois group $G := \text{Gal}(\tilde{K}/\mathbf{Q})$.
 - (b) Show that $G \cong D_4$. (Hint: for instance, you could find a couple generators and show they satisfy the correct relations.)
 - (c) Write down the action of G on the four conjugates of $\sqrt[4]{2}$ (for instance, you could label the four roots 1, 2, 3, 4 and write down an explicit isomorphism of G with a subgroup of S_4).
 - (d) Work out the subgroup lattice of D_4 and the corresponding lattice of intermediate extensions of \tilde{K}/\mathbf{Q} . In the latter, indicate which extensions are Galois and what their Galois groups are.
- (2) Intersections and composita in Galois extensions. Suppose we have the following diagram of field extensions:



where K/F is Galois with Galois group G and E_i corresponds to $H_i \leq G$ (i.e. $E_i = K^{H_i}$).

- (a) Show that $E_1 \cap E_2$ corresponds to $\langle H_1, H_2 \rangle$ (the subgroup of G generated by H_1 and H_2).
- (b) Show that $H_1 \cap H_2$ corresponds to $E_1 E_2$.