Assignment 12 – All 2 parts – Math 612

Due in class: Thursday, May 2nd, 2019

(1) A D_4-extension. Let $K = \mathbf{Q}(\sqrt[4]{2})$ and let $\widetilde{K} = \mathbf{Q}(\sqrt[4]{2}, i)$. For this exercise, I recommend you look back at everything we did in class with $\mathbf{Q}(\sqrt[3]{2})$. You'll probably also want to look back at problems (1) and (2) of Assignment 4 of last semester for material on dihedral groups.

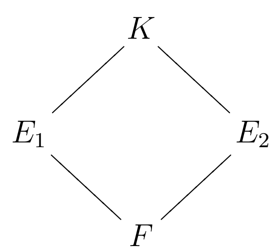
(a) Show that $[\widetilde{K} : \mathbf{Q}] = 8$. Show that $\widetilde{K}/\mathbf{Q}$ is Galois (and hence the Galois closure of $K/\mathbf{Q}$) and write down the 8 elements of the Galois group $G := \mathrm{Gal}(\widetilde{K}/\mathbf{Q})$.

(b) Show that $G \cong D_4$. (Hint: for instance, you could find a couple generators and show they satisfy the correct relations.)

(c) Write down the action of G on the four conjugates of $\sqrt[4]{2}$ (for instance, you could label the four roots $1, 2, 3, 4$ and write down an explicit isomorphism of G with a subgroup of S_4).

(d) Work out the subgroup lattice of D_4 and the corresponding lattice of intermediate extensions of $\widetilde{K}/\mathbf{Q}$. In the latter, indicate which extensions are Galois and what their Galois groups are.

(2) Intersections and composita in Galois extensions. Suppose we have the following diagram of field extensions:



where K/F is Galois with Galois group G and E_i corresponds to $H_i \leq G$ (i.e. $E_i = K^{H_i}$).

(a) Show that $E_1 \cap E_2$ corresponds to $\langle H_1, H_2 \rangle$ (the subgroup of G generated by H_1 and H_2).

(b) Show that $H_1 \cap H_2$ corresponds to $E_1 E_2$.

(3) Abelian and cyclic extensions. A field extension K/F is called *abelian* (resp. *cyclic*) if it is Galois and $\mathrm{Gal}(K/F)$ is abelian (resp. cyclic). Show that any intermediate extension E/F of an abelian (resp. cyclic) extension is itself abelian (resp. cyclic).

(4) Show that if K/F is Galois with Galois group G and H is a not necessarily normal subgroup corresponding to the intermediate extension E, then there is a natural bijection between the set of F-embeddings of E into $\overline{K}$ and the set of cosets G/H.

(5) Let $f(x)$ be a separable polynomial with roots $\alpha_1, \dots, \alpha_n$. Show that

$$\Delta(f) = (-1)^{\binom{n}{2}} \prod_{i=1}^{n} f'(\alpha_i).$$