Assignment 1 – Part 1 – Math 612

(1) Let R be a ring and let M be an R-module. Denote by $h_M : R\text{-Mod} \to \text{Ab}$ the covariant Hom functor $h_M(N) = \text{Hom}_R(M,N)$ and let h^M be the contravariant Hom functor $h^M(N) = \text{Hom}_R(N,M)$.

(a) Show that h_M is an additive functor.

(b) A contravariant functor $\mathcal{F} : R\text{-Mod} \to \text{Ab}$ is called *additive* if the map

$$\text{Hom}_R(M,N) \to \text{Hom}_{\text{Ab}}(\mathcal{F}(N), \mathcal{F}(M))$$

sending f to $\mathcal{F}(f)$ is a group homomorphism. A contravariant additive functor $\mathcal{F}$ is called *left exact* if for every short exact sequence

$$0 \to A \to B \to C \to 0,$$

the sequence

$$0 \to \mathcal{F}(C) \to \mathcal{F}(B) \to \mathcal{F}(A)$$

is exact. A contravariant additive functor $\mathcal{F}$ is called *right exact* if for every short exact sequence

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the sequence

$$\mathcal{F}(C) \to \mathcal{F}(B) \to \mathcal{F}(A) \to 0$$

is exact. Show that h^M is a left exact additive functor.

(c) Show that in general h_M is not right exact by considering $R = \mathbf{Z}$, $M = \mathbf{Z}/2\mathbf{Z}$, and the short exact sequence

$$0 \to 2\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

(d) Show that in general h^M is not right exact by considering $R = \mathbf{Z}$, $M = \mathbf{Z}$, and the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0.$$

(Hint: you can show that $\text{Hom}_{\mathbf{Z}}(\mathbf{Q},\mathbf{Z}) = 0$, but that $\text{Hom}_{\mathbf{Z}}(\mathbf{Z},\mathbf{Z}) \neq 0$.) (Remark: this same short exact sequence can be used in the previous part, as well.)