Assignment 1 – All 2 parts – Math 612

Due in class: Thursday, Jan. 17, 2019

(1) Let R be a ring and let M be an R-module. Denote by $h_M : R\text{-Mod} \to \text{Ab}$ the covariant Hom functor $h_M(N) = \text{Hom}_R(M, N)$ and let h^M be the contravariant Hom functor $h^M(N) = \text{Hom}_R(N, M)$.

(a) Show that h_M is an additive functor.

(b) A contravariant functor $\mathcal{F} : R\text{-Mod} \to \text{Ab}$ is called *additive* if the map

$$\text{Hom}_R(M, N) \to \text{Hom}_{\text{Ab}}(\mathcal{F}(N), \mathcal{F}(M))$$

sending f to $\mathcal{F}(f)$ is a group homomorphism. A contravariant additive functor $\mathcal{F}$ is called *left exact* if for every short exact sequence

$$0 \to A \to B \to C \to 0,$$

the sequence

$$0 \to \mathcal{F}(C) \to \mathcal{F}(B) \to \mathcal{F}(A)$$

is exact. A contravariant additive functor $\mathcal{F}$ is called *right exact* if for every short exact sequence

$$0 \to A \to B \to C \to 0,$$

the sequence

$$\mathcal{F}(C) \to \mathcal{F}(B) \to \mathcal{F}(A) \to 0$$

is exact. Show that h^M is a left exact additive functor.

(c) Show that in general h_M is not right exact by considering $R = \mathbf{Z}$, $M = \mathbf{Z}/2\mathbf{Z}$, and the short exact sequence

$$0 \to 2\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

(d) Show that in general h^M is not right exact by considering $R = \mathbf{Z}$, $M = \mathbf{Z}$, and the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0.$$

(Hint: you can show that $\text{Hom}_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Z}) = 0$, but that $\text{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}) \neq 0$.) (Remark:

this same short exact sequence can be used in the previous part, as well.)

(2) Let R be an integral domain. Show that every torsionfree divisible R-module is injective. (Hint: try to mimic how we extended a given $\varphi : I \to E$ to $R \to E$ in class when we considered R being a PID.)

(3) Show that $\mathbf{Z}/6\mathbf{Z}$ is injective as a module over itself.

(4) Prove each of the following.

   (a) If M is any module and E is a submodule that is injective, then E is a direct summand of M.

   (b) Every direct summand of an injective module E is itself injective.

   (c) For each i in some index set I, suppose E_i is a module. Show that $\prod_{i \in I} E_i$ is injective if and only if E_i is injective for all $i \in I$.