Assignment 2 – All 2 parts – Math 612

Due in class: Thursday, Jan. 24, 2019

(1) Suppose $\mathcal{C}$ is a category with a zero object (so that one can define kernels and cokernels). Show that every kernel is a monomorphism and every cokernel is an epimorphism.

(2) Prove the following statements given in class.

  (a) Suppose $\mathcal{C}$ is an additive category and $f : A \to B$ is a morphism in $\mathcal{C}$ whose kernel exists. Show that f is monic if and only if its kernel is 0 (i.e. its kernel is the equivalence class of the zero morphism $0 : 0 \to A$.)

  (b) Suppose $\mathcal{C}$ is an additive category and $g : B \to C$ is a morphism in $\mathcal{C}$ whose cokernel exists. Show that g is epic if and only if its cokernel is 0 (i.e. its cokernel is the equivalence class of the zero morphism $0 : C \to 0$.)

  (c) Show that if f is a morphism in an abelian category, then $\mathrm{im}(f) = \ker(\mathrm{coker}(f))$.

(3) Let R and S be rings and let $\mathcal{F} : R\text{-Mod} \to S\text{-Mod}$ be an exact additive covariant functor.

  (a) Suppose $\varphi : A \to B$ is an R-module homomorphism. Show that $\mathcal{F}(\ker(\varphi)) \cong \ker(\mathcal{F}(\varphi))$, $\mathcal{F}(\mathrm{coker}(\varphi)) \cong \mathrm{coker}(\mathcal{F}(\varphi))$, and $\mathcal{F}(\mathrm{im}(\varphi)) \cong \mathrm{im}(\mathcal{F}(\varphi))$, i.e $\mathcal{F}$ commutes with kernels, cokernels, and images.

  (b) Show that $\mathcal{F}$ induces a functor $\mathrm{Kom}(R) \to \mathrm{Kom}(S)$ (that we'll also denote by $\mathcal{F}$) given by $(\mathcal{F}(A^\bullet))^n = \mathcal{F}(A^n)$ with differential $(\mathcal{F}(d))^n = \mathcal{F}(d^n)$, and if φ is a morphism of complexes, then $\mathcal{F}(\varphi)_n = \mathcal{F}(\varphi_n)$. ($\mathcal{F}$ need not be exact for this part, so do not use this property.)

  (c) Show that $\mathcal{F}$ commutes with homology, i.e. if $(A^\bullet, d^\bullet)$ is any complex of R-modules, then $H^n(\mathcal{F}(A^\bullet)) = \mathcal{F}(H^n(A^\bullet))$ for all $n \in \mathbf{Z}$.