Assignment 3 – Part 1 – Math 612

(1) $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$. In this exercise, you'll compute the $\mathbf{Z}$-module $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$ for $m, n \in \mathbf{Z}_{\geq 2}$. Recall that I said we will define $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, -)$ as the first right derived functor of $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, -)$ and hence it can be computed by taking an injective resolution of $\mathbf{Z}/n\mathbf{Z}$.

(a) Show that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}/n\mathbf{Z} & \longrightarrow & \mathbf{Q}/\mathbf{Z} & \xrightarrow{d^0} & \mathbf{Q}/\mathbf{Z} & \xrightarrow{d^1} & 0 \\ & & 1 & \longmapsto & \frac{1}{n} + \mathbf{Z} & & & & \\ & & & & a + \mathbf{Z} & \longmapsto & na + \mathbf{Z} & & \end{array}$$

is an injective resolution of $\mathbf{Z}/n\mathbf{Z}$.

(b) Show that $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \cong (\frac{1}{m}\mathbf{Z})/\mathbf{Z} \cong \mathbf{Z}/m\mathbf{Z}$.

(c) Applying $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, -)$ to the complex

$$0 \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow 0$$

you get the complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \xrightarrow{d^0_*} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \xrightarrow{d^1_*} 0.$$

Recall that $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}) := \ker d^1_* / \operatorname{im} d^0_*$. Show that

$$\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}) \cong (\mathbf{Z}/m\mathbf{Z})\big/(n(\mathbf{Z}/m\mathbf{Z})) \cong \mathbf{Z}/\gcd(m,n)\mathbf{Z}.$$