Assignment 3 – All 2 parts – Math 612

Due in class: Thursday, Jan. 31, 2019

(1) $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$. In this exercise, you'll compute the $\mathbf{Z}$-module $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z})$ for $m, n \in \mathbf{Z}_{\geq 2}$. Recall that I said we will define $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, -)$ as the first right derived functor of $\operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, -)$ and hence it can be computed by taking an injective resolution of $\mathbf{Z}/n\mathbf{Z}$.

(a) Show that

$$0 \longrightarrow \mathbf{Z}/n\mathbf{Z} \longrightarrow \mathbf{Q}/\mathbf{Z} \xrightarrow{d^0} \mathbf{Q}/\mathbf{Z} \xrightarrow{d^1} 0$$

$$1 \longmapsto \tfrac{1}{n} + \mathbf{Z}$$

$$a + \mathbf{Z} \longmapsto na + \mathbf{Z}$$

is an injective resolution of $\mathbf{Z}/n\mathbf{Z}$.

(b) Show that $\operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \cong (\frac{1}{m}\mathbf{Z})/\mathbf{Z} \cong \mathbf{Z}/m\mathbf{Z}$.

(c) Applying $\operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, -)$ to the complex

$$0 \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow 0$$

you get the complex

$$0 \longrightarrow \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \xrightarrow{d^0_*} \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \xrightarrow{d^1_*} 0.$$

Recall that $\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}) := \ker d^1_* / \operatorname{im} d^0_*$. Show that

$$\operatorname{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}) \cong (\mathbf{Z}/m\mathbf{Z})\big/(n(\mathbf{Z}/m\mathbf{Z})) \cong \mathbf{Z}/\gcd(m,n)\mathbf{Z}.$$

(2) **The snake lemma.** In class, we considered the commutative diagram for R-modules

$$\begin{array}{ccccccc} & & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_3} & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{f'} & N_2 & \xrightarrow{g'} & N_3 & & \end{array}$$

with exact rows and said that we obtain an exact sequence

$$\ker d_1 \longrightarrow \ker d_2 \longrightarrow \ker d_3 \xrightarrow{\delta} \operatorname{coker} d_1 \longrightarrow \operatorname{coker} d_2 \longrightarrow \operatorname{coker} d_3.$$

(a) In class, we showed exactness at $\ker d_2$ and $\ker d_3$. Show exactness at $\operatorname{coker} d_1$, and $\operatorname{coker} d_2$.

(b) Show that if $M_1 \to M_2$ is injective, then so is $\ker d_1 \to \ker d_2$. Show that if $N_2 \to N_3$ is surjective, then so is $\operatorname{coker} d_2 \to \operatorname{coker} d_3$.

(3) **The five lemma.** Consider the commutative diagram of R-modules

$$\begin{array}{ccccccccc} M_1 & \xrightarrow{d_1} & M_2 & \xrightarrow{d_2} & M_3 & \xrightarrow{d_3} & M_4 & \xrightarrow{d_4} & M_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ N_1 & \xrightarrow{d_1'} & N_2 & \xrightarrow{d_2'} & N_3 & \xrightarrow{d_3'} & N_4 & \xrightarrow{d_4'} & N_5 \end{array}$$

whose rows are exact.

(a) Suppose f_1 is surjective and f_2 and f_4 are injective. Show that f_3 is injective.

(b) Suppose f_5 is injective and f_2 and f_4 are surjective. Show that f_3 is surjective.

(c) This is a remark. These two parts combine to give the "five lemma": if f_1 is surjective, f_5 is injective, and f_2 and f_4 are isomorphisms, then f_3 is an isomorphism.