Assignment 4 – Part 1 – Math 612

(1) Higher Ext.

(a) Let R be a PID. Show that every R-module has an injective resolution of length 1 (the *length* of a resolution $0 \to M \to E^0 \to E^1 \to \cdots$ is the maximum n such that $E^n \neq 0$). (Hint: first embed the module M into an injective module to get E^0, then take E^1 to be the cokernel, and show this works. Recall that injective and divisible are the same over a PID.)

(b) Conclude that if R is a PID and A and B are any R-modules, then $\mathrm{Ext}^n_R(A, B) = 0$ for all $n \geq 2$.

(c) Show that if R is in fact a field F, then every F-module is injective, so that it has an injective resolution of length 0. Conclude that $\mathrm{Ext}^n_F(A, B) = 0$ for all $n \geq 1$ and all F-vector spaces A and B.

(d) This is just a remark. Given a ring R, one defines the (*left*) *global dimension of* R, gl dim R, to be the maximum n such that there exists (left) R-modules A and B with $\mathrm{Ext}^n(A, B) \neq 0$ (it's also called the *global homological dimension* or simply the *homological dimension* of R). The above exercises show that the global dimension of a field F is 0 and that of a PID (that isn't a field), such as $F[x]$, is 1. One can in fact show that gl dim $F[x_1, \ldots, x_n] = n$ for a field F. This is meant to reflect the fact that the ring $F[x_1, \ldots, x_n]$ is the ring of polynomial functions on the n-dimensional space F^n. The more standard definition of dimension of a commutative ring is that of its Krull dimension (which we will see later this semester). These two notions may not coincide.