Assignment 4 – All 2 parts – Math 612

Due in class: Thursday, Feb. 7, 2019

(1) Higher Ext.

(a) Let R be a PID. Show that every R-module has an injective resolution of length 1 (the *length* of a resolution $0 \to M \to E^0 \to E^1 \to \cdots$ is the maximum n such that $E^n \neq 0$). (Hint: first embed the module M into an injective module to get E^0, then take E^1 to be the cokernel, and show this works. Recall that injective and divisible are the same over a PID.)

(b) Conclude that if R is a PID and A and B are any R-modules, then $\mathrm{Ext}^n_R(A, B) = 0$ for all $n \geq 2$.

(c) Show that if R is in fact a field F, then every F-module is injective, so that it has an injective resolution of length 0. Conclude that $\mathrm{Ext}^n_F(A, B) = 0$ for all $n \geq 1$ and all F-vector spaces A and B.

(d) This is just a remark. Given a ring R, one defines the (*left*) *global dimension of* R, gl dim R, to be the maximum n such that there exists (left) R-modules A and B with $\mathrm{Ext}^n(A, B) \neq 0$ (it's also called the *global homological dimension* or simply the *homological dimension* of R). The above exercises show that the global dimension of a field F is 0 and that of a PID (that isn't a field), such as $F[x]$, is 1. One can in fact show that gl dim $F[x_1, \ldots, x_n] = n$ for a field F. This is meant to reflect the fact that the ring $F[x_1, \ldots, x_n]$ is the ring of polynomial functions on the n-dimensional space F^n. The more standard definition of dimension of a commutative ring is that of its Krull dimension (which we will see later this semester). These two notions may not coincide.

(2) More Ext computations.

(a) Let m be a positive integer. Use the projective resolution

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/m\mathbf{Z} \to 0$$

(where the first arrow is multiplication by m and the second one is the usual quotient map) of $\mathbf{Z}/m\mathbf{Z}$ as a $\mathbf{Z}$-module to show that for any $\mathbf{Z}$-module N,

$$\mathrm{Ext}^0_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, N) = N[m], \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, N) = N/mN, \text{ and } \mathrm{Ext}^i(\mathbf{Z}/m\mathbf{Z}, N) = 0 \text{ for all } i \geq 2,$$

where $N[m] := \{n \in N : mn = 0\}$ is the *m-torsion* of N.

(b) Let p be a prime. First show that $\mathbf{Z}/p^2\mathbf{Z}$ is injective as a module over itself. Let $\iota : \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p^2\mathbf{Z}$ be the inclusion sending 1 to p. Now, show that

$$0 \to \mathbf{Z}/p\mathbf{Z} \xrightarrow{\iota} \mathbf{Z}/p^2\mathbf{Z} \xrightarrow{p} \mathbf{Z}/p^2\mathbf{Z} \xrightarrow{p} \mathbf{Z}/p^2\mathbf{Z} \xrightarrow{p} \cdots$$

is an injective resolution of $\mathbf{Z}/p\mathbf{Z}$ as a $\mathbf{Z}/p^2\mathbf{Z}$-module (where the maps denoted by p are multiplication by p maps). Conclude that

$$\mathrm{Ext}^n_{\mathbf{Z}/p^2\mathbf{Z}}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z}) \cong \mathbf{Z}/p\mathbf{Z} \text{ for all } n.$$

(3) Let R be a ring.

(a) Show that an R-module E is injective if and only if $\mathrm{Ext}^n(M, E) = 0$ for all $n \geq 1$ and all R-modules M.

(b) Show that an R-module P is projective if and only if $\mathrm{Ext}^n(P, N) = 0$ for all $n \geq 1$ and all R-modules N.