Assignment 5 – Part 1 – Math 612

(1) Suppose

$$0 \longrightarrow N \xrightarrow{e} E^0 \xrightarrow{e^0} E^1 \xrightarrow{e^1} \cdots$$

is an injective resolution of N. Show that for all $p \geq 1$

$$\mathrm{Ext}_R^{p+n}(M, N) \cong \mathrm{Ext}_R^p(M, \ker e^n).$$

(Hint: Dimension shifting.)

(2) Yoneda Ext. A concrete interpretation of $\mathrm{Ext}^n(M, N)$ was given by Yoneda as follows (the case of $n = 1$ was already known). Let M and N be R-modules (or two objects in an abelian category which need not have enough injectives). An *extension of M by N* is a short exact sequence ξ

$$0 \to N \to X \to M \to 0$$

(one sometimes refers to X as an extension of M by N). More generally, for a positive integer n, an *n-fold extension of M by N* (or a *degree N extension of M by N*) is an exact sequence ξ

$$0 \longrightarrow N \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} \cdots \longrightarrow X_1 \xrightarrow{f_1} M \longrightarrow 0.$$

Let's write this as

$$\xi : 0 \longrightarrow N \longrightarrow (X_\bullet, f_\bullet) \longrightarrow M \longrightarrow 0.$$

Given another n-fold extension of M by N

$$\chi : 0 \longrightarrow N \longrightarrow (Y_\bullet, g_\bullet) \longrightarrow M \longrightarrow 0,$$

a *morphism* $\pi : \xi \to \chi$ is a collection of maps $\pi_k : X_k \to Y_k$ making the following diagram commute

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & N & \xrightarrow{f_{n+1}} & X_n & \longrightarrow & \cdots & \longrightarrow & X_1 & \xrightarrow{f_1} & M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \pi_n} & & & & \downarrow{\scriptstyle \pi_1} & & \| & & \\
0 & \longrightarrow & N & \xrightarrow{g_{n+1}} & Y_n & \longrightarrow & \cdots & \longrightarrow & Y_1 & \xrightarrow{g_1} & M & \longrightarrow & 0.
\end{array}$$

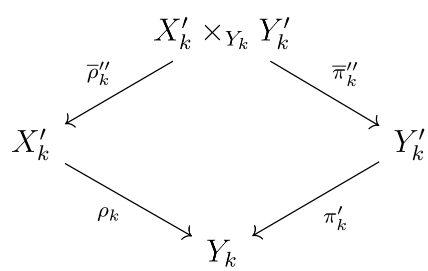
(a) Define a relation on the set of n-fold extensions of M by N as follows. Say ξ *is* *equivalent to* χ, written $\xi \sim \chi$ if there's yet another n-fold extension

$$\xi' : 0 \longrightarrow N \longrightarrow (X'_\bullet, f'_\bullet) \longrightarrow M \longrightarrow 0.$$

and two morphisms $\pi : \xi' \to \xi$ and $\rho : \xi' \to \chi$. Show that this is an equivalence relation. Hint: The tough part here is transitivity. Suppose you have yet another extension $\psi = (Z_\bullet, h_\bullet)$ and $\chi \sim \psi$ via an extension $\chi' = (Y'_\bullet, g'_\bullet)$ with maps $\pi' : \chi' \to \chi$ and $\rho' : \chi' \to \psi$, show that you can get that $\xi \sim \psi$ via an extension $(Z'_\bullet, h'_\bullet)$ where $Z'_k := X'_k \times_{Y_k} Y'_k$, the fibre product of X'_k with Y'_k over Y_k. Recall that the fibre product in the category of R-modules is given by

$$X'_k \times_{Y_k} Y'_k = \{(x, y) \in X'_k \times Y'_k : \rho_k(x) = \pi'_k(y)\}$$

equipped with the natural coordinate projection maps $\overline{\rho}''_k : X'_k \times_{Y_k} Y'_k \to X'_k$ and $\overline{\pi}''_k : X'_k \times_{Y_k} Y'_k \to Y'_k$ as illustrated in this commutative diagram:



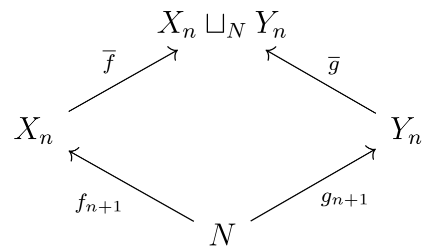
(b) We will now denote the set of equivalences classes of n-fold extensions of M by N by $\mathrm{YExt}^n_R(M, N)$. We will be able to drop the 'Y' from the notation eventually because, in fact, there is a group law on this set given by the so-called Baer sum and a natural isomorphism with the abelian group $\mathrm{Ext}^n_R(M, N)$. In this part, you'll prove a little bit about the group law. Suppose ξ and ξ' are two n-fold extensions of M by N. Define their *Baer sum* to be the extension

$$\xi \oplus \xi' : 0 \longrightarrow N \longrightarrow (\widetilde{X}_\bullet, \widetilde{f}_\bullet) \longrightarrow M \longrightarrow 0,$$

where

$$\widetilde{X}_k := \begin{cases} X_1 \times_M Y_1 & k = 1 \\ X_n \sqcup_N Y_n & k = n \\ X_k \oplus Y_k & \text{otherwise.} \end{cases}$$

Here, $X_n \sqcup_N Y_n$ denotes the fibre coproduct of X_n and Y_n over N. Recall that the fibre coproduct in the category of R-modules lives in the diagram



and is given explicitly by

$$X_n \sqcup_N Y_n := (X_n \oplus Y_n)/\{(f_{n+1}(z), -g_{n+1}(z)) : z \in N\}.$$

Now, what needs to be shown is this is well-defined independently of the choice of representatives of the equivalence classes, but I won't make you do that because it sounds terrible. Instead, just show that the sequence $\xi \oplus \xi'$ is indeed an n-fold extension of M by N (i.e. that the sequence is exact).

(c) Now, let's define a map $\theta : \mathrm{YExt}^n_R(M,N) \to \mathrm{Ext}^n_R(M,N)$ that will be the desired natural group isomorphism. First off, let

$$0 \longrightarrow N \xrightarrow{\ e\ } E^0 \xrightarrow{\ e^0\ } E^1 \xrightarrow{\ e^1\ } \cdots$$

be an injective resolution of N. Since an extension ξ is an exact sequence, a result proved in class says that there is a unique (up to homotopy) morphism of complexes φ from ξ to $0 \to N \to E^\bullet$ with $\varphi_{-1} = \mathrm{id}_N$, i.e. a unique (up to homotopy) collection of morphisms φ_k such that

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & N & \xrightarrow{f_{n+1}} & X_n & \xrightarrow{f_n} & \cdots & \longrightarrow & X_1 & \xrightarrow{f_1} & M & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \| & & \downarrow{\scriptstyle \varphi_0} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} & & \\
0 & \longrightarrow & N & \xrightarrow{e} & E^0 & \xrightarrow{e^0} & \cdots & \xrightarrow{e^{n-2}} & E^{n-1} & \xrightarrow{e^{n-1}} & E^n & \xrightarrow{e^n} & E^{n+1} & \xrightarrow{e^{n+1}} & \cdots
\end{array}$$

First, show that $\mathrm{im}\,\varphi_n \subseteq \mathrm{im}\,e^{n-1}$, so that φ_n induces a map $\beta_\xi : M \to \mathrm{im}\,e^{n-1}$. Now, consider the short exact sequence

$$0 \to \ker e^{n-1} \to E^{n-1} \to \mathrm{im}\,e^{n-1} \to 0$$

Apply $\mathrm{Hom}_R(M, -)$ to this sequence and use the long exact sequence and the result from Question (1) to show that you get a natural map

$$\delta_\xi : \mathrm{Hom}_R(M, \operatorname{im} e^{n-1}) \to \mathrm{Ext}^n_R(M, N).$$

The map θ is then defined by setting $\theta(\xi) := \delta_\xi(\beta_\xi)$. Of course, you'd now have to show this is a well-defined group isomorphism, but this question is already long enough. Anyway, this definition of θ is a nice example of how one gets interesting natural maps via connecting homomorphisms δ.