Assignment 6 – All 2 parts – Math 612

Due in class: Thursday, Feb. 21, 2019

(1) Let R be a commutative ring. Recall from class that if M is a left (or right) R-module, equipped with its standard bimodule structure, and N is a left R-module, then $M \otimes_R N$ is naturally a left R-module via

$$r \cdot (m \otimes n) = (rm) \otimes n.$$

Also recall that if M, N, and X are left R-modules, then a function $\psi : M \times N \to X$ is called R-*bilinear* if it is R-biadditive and

$$r\psi(m, n) = \psi(rm, n) = \psi(m, rn)$$

for all $r \in R$, $m \in M$, and $n \in N$.

(a) Show that when R is commutative, the natural map $\psi_{\text{univ}} : M \times N \to M \otimes_R N$ is R-linear.

(b) Show that when R is commutative, $M \otimes_R N$ satisfies the following universal property: for every R-module X and every R-bilinear map $\psi : M \times N \to X$, there is a unique R-linear map φ such that the diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\psi_{\text{univ}}} & M \otimes_R N \\ & \searrow^{\psi} & \downarrow{\scriptstyle \exists! \varphi} \\ & & X \end{array}$$

commutes.

(2) Let R be a ring, let $f : A \to A'$ and $f' : A' \to A''$ be homomorphisms of right R-modules, and let $g : B \to B'$ and $g' : B' \to B''$ be homomorphisms of left R-modules. Show that $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$.

(3) Fix a right R-module M. Show that the association sending $N \mapsto M \otimes_R N$ (for N a left R-module) and $\varphi \mapsto \text{id}_M \otimes \varphi$ (for $\varphi \in \text{Hom}_R(N, N')$) is a covariant additive functor from the category of left R-modules to the category of abelian groups. Furthermore, show that if M is in fact an (S, R)-bimodule, then it is a covariant additive functor from R-Mod to S-Mod.