Assignment 7 – All 2 parts – Math 612

Due in class: Thursday, Feb. 28, 2019

(1) Let $\varphi : R \to S$ be a homomorphism of rings and let $\mathcal{F}$ be the forgetful functor from S-Mod to R-Mod. We showed in class that $\mathcal{F}$ has a left adjoint, namely $S \otimes_R -$. Show that $\mathrm{Hom}_R(S, -)$ is a right adjoint to $\mathcal{F}$. (Remark: thus $\mathcal{F}$ has both a left and a right adjoint; the left adjoint is sometimes called *induction (from R to S)* while the right adjoint is sometimes called *coinduction (from R to S)*. They are sometimes isomorphic, but other times not.)

(2) Let G be a group and let G-**Set** be the category of G-sets (i.e. the objects are sets equipped with a left action of G and the morphisms are the G-equivariant functions between the sets). Let $\mathcal{F} : G\text{-}\mathbf{Set} \to \mathbf{Set}$ be the forgetful functor.

(a) Given a set X, define $\mathrm{Ind}^G(X)$ to be the set $G \times X$ with the action $\gamma \cdot (g, x) := (\gamma g, x)$. Show that Ind^G is a left adjoint to $\mathcal{F}$.

(b) Given a set Y, define $\mathrm{coInd}^G(Y)$ to be the set of functions $\mathrm{Hom}_{\mathbf{Set}}(G, Y)$ with the action $(\gamma \cdot \psi)(g) := \psi(g\gamma)$. Show that coInd^G is a right adjoint to $\mathcal{F}$.

(3) Let R and S be rings. Suppose M is a right S-module, P is an (S, R)-bimodule, and N is a left R-module.

(a) Show that the tensor product is associative, i.e.

$$M \otimes_S (P \otimes_R N) \cong (M \otimes_S P) \otimes_R N$$

as $\mathbf{Z}$-modules.

(b) Suppose R is commutative and S is an R-algebra (note that S is then naturally an (S, R)-bimodule). Show that for every left R-module N and right S-module M

$$M \otimes_S (S \otimes_R N) \cong M \otimes_R N$$

as $\mathbf{Z}$-modules.

(c) Suppose again that R is commutative and S is an R-algebra. Show that if N is a flat left R-module, then $S \otimes_R N$ is a flat left S-module.