Assignment 8 – All 2 parts – Math 612

Due in class: Thursday, Mar. 28, 2019

(1) Let K/F be an algebraic field extension.

(a) Let $\{\alpha_i : i \in I\} \subseteq K$. Show that $F(\{\alpha_i : i \in I\}) = \{f((\alpha_i)_{i \in I}) : f \in F[I]\}$, i.e. every element of $F(\{\alpha_i : i \in I\})$ is simply a polynomial in the α_i with coefficients in F (i.e. you don't need to take fractions of polynomials).

(b) Let K_1 and K_2 be two extensions of F contained in K. Suppose $\{v_n : n \in N\}$ is a basis of K_1 (as an F-vector space) and $\{w_m : m \in M\}$ is a basis of K_2 (as an F-vector space). Show that the compositum is spanned by $\{v_n w_m : n \in N, m \in M\}$ (as an F-vector space). (Hint: one has that $K_1 = F(\{v_n : n \in N\})$ and similarly for K_2.)

(c) Conclude that $[K_1 K_2 : F] \leq [K_1 : F][K_2 : F]$. When the $[K_i : F]$ are finite and relatively prime, show that equality holds.

(d) Suppose $f(x)$ and $g(x)$ are irreducible in $F[x]$. Let α be a root of $f(x)$. If the degrees of f and g are relatively prime, show that $g(x)$ is irreducible in $F(\alpha)[x]$.

(2) Suppose K/F is an algebraic extension. Show that it is the compositum of all its finite sub extensions (i.e. of all $k \leq K$ such that k is a finite extension of F).

(3) Let Ω/F be a field extension. Show that Ω is an algebraic closure of F if and only if it is a maximal algebraic extension of F.