Assignment 3 – Part 1 – Math 661

(1) The Chinese Remainder Theorem in Dedekind Domains. Let $\mathcal{O}$ be a Dedekind Domain.

(a) Let I be an ideal of $\mathcal{O}$ and suppose

$$I = \prod_{i=1}^{r} \mathfrak{p}^{e_i}$$

is the prime factorization of I (with the $\mathfrak{p}_i$ distinct). Fix an $\alpha \in \mathcal{O}$. Show that some $\beta \in \mathcal{O}$ is such that $\beta \equiv \alpha \pmod{I}$ if and only if $\beta \equiv \alpha \pmod{\mathfrak{p}_i^{e_i}}$ for all i.

(b) For $i = 1, \ldots, r$, let $\mathfrak{p}_i$ be distinct non-zero prime ideals of $\mathcal{O}$, let $e_i \in \mathbf{Z}_{\geq 1}$, and fix $\alpha_i \in \mathcal{O}$. Show that there is an element $\alpha \in \mathcal{O}$ such that

$$\alpha \equiv \alpha_i \pmod{\mathfrak{p}_i^{e_i}}, \text{ for } i = 1, 2, \ldots, r.$$

(Hint: you can adapt the proof for $\mathbf{Z}$ given what we know about the divisibility theory of ideals of a Dedekind domain.)

(c) Explain how this proves the following (called the Chinese Remainder Theorem): For $j = 1, \ldots, r$, let I_j be pairwise relatively prime ideals of $\mathcal{O}$ and let α_j be fixed elements of $\mathcal{O}$. Then there is an element $\alpha \in \mathcal{O}$ such that

$$\alpha \equiv \alpha_j \pmod{I_j}, \text{ for } j = 1, 2, \ldots, r.$$

(2) (a) Let R be a (commutative) ring and suppose I and J are ideals of R generated by S and T, respectively. Show that IJ is generated by $\{st : s \in S, t \in T\}$.

(b) Let $K = \mathbf{Q}(\sqrt{79})$ so that $\mathcal{O}_K = \mathbf{Z}[\sqrt{79}]$. We saw in class that every ideal of $\mathcal{O}_K$ can be generated by 2 elements. Let $I = (3, 2+\sqrt{79})$. Show that $I^2 = (9, 5+\sqrt{79})$.

(c) With K and I as in part (b), let $J = (3, 1+\sqrt{79})$. Show that IJ is principal!

(d) Bonus: show that the ideals I, J, and I^2 are not principal. (Hint: try looking at their norms and the possible norms of elements of $\mathcal{O}_K$.)