Assignment 4 – Part 1 – Math 661

(1) Let $\omega = e^{2\pi i/3} = \frac{-1}{2} + \frac{\sqrt{-3}}{2}$ and let $K = \mathbf{Q}(\sqrt[3]{2}, \omega)$ (the Galois closure of $\mathbf{Q}(\sqrt[3]{2})$).

(a) Determine the factorization types of $2, 3, 5$, and 7 in K. (Hint: you can use what you figured out in Question (6b) of the previous assignment together with what you know about factorization in the quadratic subfield $\mathbf{Q}(\omega)$ of K.)

(b) Recall that $K/\mathbf{Q}$ is Galois with Galois group isomorphic to S_3, and that S_3 has subgroups isomorphic to $1, C_2, C_3$, and S_3. For each of $p = 2, 3, 5$, and 7 and every prime $\mathfrak{P}$ of $\mathcal{O}_K$ dividing p, determine the decomposition group and inertia group of $\mathfrak{P}$ (up to isomorphism, i.e. which of $1, C_2, C_3$, or S_3 is each of these).

(2) Let $\alpha = \dfrac{\sqrt{2} + \sqrt{6}}{2}$ and let $K = \mathbf{Q}(\alpha)$.

(a) Show that $K = \mathbf{Q}(\sqrt{2}, \sqrt{3})$.

(b) Determine the minimal polynomial $f_\alpha(x)$ of α and show that α is integral over $\mathbf{Z}$.

(c) Show that $2 \nmid [\mathcal{O}_K : \mathbf{Z}[\alpha]]$. (Hint: consider $f_\alpha(x - 1)$.)

(d) Show that $\mathcal{O}_K = \mathbf{Z}[\alpha]$ as follows (or by some other means). First, compute the discriminant of $\mathbf{Z}[\alpha]$. Then, using the fact that K is the compositum of $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$ and what we know about discriminants of composita, conclude using the previous part that no odd prime divides $[\mathcal{O}_K : \mathbf{Z}[\alpha]]$ either.

(e) Determine the decomposition groups of all primes above 2 and 3 in K.