Assignment 4 – All 4 parts – Math 661

Due in class: Thursday, Mar. 12, 2020

(1) Let $\omega = e^{2\pi i/3} = \frac{-1}{2} + \frac{\sqrt{-3}}{2}$ and let $K = \mathbf{Q}(\sqrt[3]{2}, \omega)$ (the Galois closure of $\mathbf{Q}(\sqrt[3]{2})$).

(a) Determine the factorization types of $2, 3, 5$, and 7 in K. (Hint: you can use what you figured out in Question (6b) of the previous assignment together with what you know about factorization in the quadratic subfield $\mathbf{Q}(\omega)$ of K.)

(b) Recall that $K/\mathbf{Q}$ is Galois with Galois group isomorphic to S_3, and that S_3 has subgroups isomorphic to $1, C_2, C_3$, and S_3. For each of $p = 2, 3, 5$, and 7 and every prime $\mathfrak{P}$ of $\mathcal{O}_K$ dividing p, determine the decomposition group and inertia group of $\mathfrak{P}$ (up to isomorphism, i.e. which of $1, C_2, C_3$, or S_3 is each of these).

(2) Let $\alpha = \dfrac{\sqrt{2} + \sqrt{6}}{2}$ and let $K = \mathbf{Q}(\alpha)$.

(a) Show that $K = \mathbf{Q}(\sqrt{2}, \sqrt{3})$.

(b) Determine the minimal polynomial $f_\alpha(x)$ of α and show that α is integral over $\mathbf{Z}$.

(c) Show that $2 \nmid [\mathcal{O}_K : \mathbf{Z}[\alpha]]$. (Hint: consider $f_\alpha(x - 1)$.)

(d) Show that $\mathcal{O}_K = \mathbf{Z}[\alpha]$ as follows (or by some other means). First, compute the discriminant of $\mathbf{Z}[\alpha]$. Then, using the fact that K is the compositum of $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$ and what we know about discriminants of composita, conclude using the previous part that no odd prime divides $[\mathcal{O}_K : \mathbf{Z}[\alpha]]$ either.

(e) Determine the decomposition groups of all primes above 2 and 3 in K.

(3) Suppose K/F is a Galois degree 4 extension of number fields with Galois group C_4. Then there is a unique intermediate extension K_2 of degree 2 over F. Suppose $\mathfrak{p}$ is a prime of F whose ramification index in K is 2. Show that $\mathfrak{p}$ is unramified in the unique intermediate quadratic extension. Conclude that every prime of K_2 that is ramified over F ramifies in K/K_2.

(4) Determine the inertia subgroups of the primes listed in Question (2e).

(5) Let A be a Dedekind domain, let $F = \mathrm{Frac}(A)$, and consider a tower of extensions

$$\begin{array}{c} K \\ | \\ K' \\ | \\ F \end{array}$$

where K/F is Galois. Let $\mathcal{O}$ be the integral closure of A in K (and $\mathcal{O}'$ the

integral closure of A in K'). Suppose $\mathfrak{P}$ is a prime ideal of $\mathcal{O}$ and let $\mathfrak{p} = \mathfrak{P} \cap A$ (and $\mathfrak{P}' = \mathfrak{P} \cap \mathcal{O}'$). Let K^{dec} be the decomposition field of $\mathfrak{P}|\mathfrak{p}$. Show that if $e(\mathfrak{P}'|\mathfrak{p}) = f(\mathfrak{P}'|\mathfrak{p}) = 1$, then $K' \subseteq K^{\mathrm{dec}}$.

(6) Let p be a prime number and let $K = \mathbf{Q}(p^{1/p}, \zeta_p)$. Show that p is totally ramified in K.

(7) Determine the factorization types of the primes $p = 2, 3, 5, 7$ in $\mathbf{Q}(\zeta_n)$, where $n = 4 \cdot 9 \cdot 5$.

(8) Consider a tower of extensions of number fields:

$$\begin{array}{c} K \\ | \\ K' \\ | \\ F \end{array}$$

with K/F Galois. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_F$, let $\mathfrak{P}$ be a prime above $\mathfrak{p}$ in $\mathcal{O}_K$, and let $\mathfrak{P}' = \mathfrak{P} \cap K'$. Suppose $\mathfrak{P}$ is unramified in K/F.

(a) Show the following relation between Artin symbols:

$$\left(\frac{K/K'}{\mathfrak{P}}\right) = \left(\frac{K/F}{\mathfrak{P}}\right)^{f(\mathfrak{P}'/\mathfrak{p})}.$$

(Hint: recall that the Artin symbol (a.k.a. the Frobenius element) is characterized by what it does to elements of $\mathcal{O}_K$ mod $\mathfrak{P}$.)

(b) Assume that K'/F is also Galois and recall that, by Galois theory, the natural surjective homomorphism $\mathrm{res} : \mathrm{Gal}(K/F) \to \mathrm{Gal}(K'/F)$ is given by 'restriction', i.e. $\mathrm{res}(\sigma) = \sigma|_{K'}$. Show that

$$\left(\frac{K'/F}{\mathfrak{P}'}\right) = \mathrm{res}\left(\frac{K/F}{\mathfrak{P}}\right).$$

(Hint: same)

(9) Recall that if K' and K'' are Galois extensions of a field F (contained in a bigger field) and $K = K'K''$ is their compositum, then K/F is Galois and $\mathrm{Gal}(K/F)$ injects into $\mathrm{Gal}(K_1/F) \times \mathrm{Gal}(K_2/F)$ via

$$\sigma \mapsto (\sigma|_{K'}, \sigma|_{K''}).$$

Now, suppose in this situation that these are extensions of number fields and let $\mathfrak{p}$ be some prime of $\mathcal{O}_F$. Show that $\mathfrak{p}$ is totally split in K/F if and only if it is totally split in both K'/F and K''/F. (Hint: consider the various Frobenii.)