Assignment 5 – Part 1 – Math 661

(1) Prime splitting in D_4-quartic fields, part I. Let K/F be a quartic extension of number fields such that the Galois closure $\widetilde{K}$ of K over F has $G := \mathrm{Gal}(\widetilde{K}/F) \cong D_4$. We call such a K/F a *D_4-quartic extension*. In order to save you some time, let me tell you a bit about D_4 and the Galois theory of D_4-quartic extensions. Recall that the group D_4 is defined to be the group of rigid motions of the plane sending a square to itself. Let ρ denote the counterclockwise rotation by $\pi/2$ and let r be any one of the reflections. Then, D_4 is generated by ρ and r and they satisfy $r\rho r = \rho^{-1}$. In particular, $D_4 = \{1, \rho, \rho^2, \rho^3, r, \rho r, \rho^2 r, \rho^3 r\}$. The non-trivial proper subgroups of D_4 are:

Order 2	$\langle \rho^2 \rangle, \langle r \rangle, \langle \rho r \rangle, \langle \rho^2 r \rangle, \langle \rho^3 r \rangle$
Order 4	$\langle \rho \rangle, \langle r, \rho^2 \rangle, \langle \rho r, \rho^2 \rangle$

Back to the field extensions. The field K has degree 4 over F and $\widetilde{K}$ has degree 8, so the subgroup H of $G = D_4$ corresponding to K has order 2. To determine which of the five order two subgroups it is, note that the first one, $\langle \rho^2 \rangle$, is the centre of D_4 and so, in particular, is a normal subgroup, so that the corresponding intermediate extension is Galois over F. But K/F is not Galois. Hence, H must be one of the 4 remaining order two subgroups. These are the order two subgroups generated by a reflection. And since we just defined r to be any reflection, we may, without loss of generality, take r to be the generator of H.

(a) First figure out the cosets in $H\backslash G$.

(b) Now, suppose $\mathfrak{p}$ is a prime of F that is unramified in $\widetilde{K}$. List the possible decomposition groups $D_{\widetilde{\mathfrak{P}}}$ for a prime $\widetilde{\mathfrak{P}}$ of $\widetilde{K}$ dividing $\mathfrak{p}$.

(c) For each of these possible $D_{\widetilde{\mathfrak{P}}}$, determine the orbits for its right action on $H\backslash G$ and the corresponding $g(\mathfrak{p})$.

(Remark: I just wanted to give you a specific example to play along with. You can take $F = \mathbf{Q}$ and $K = \mathbf{Q}(\sqrt[4]{2})$. Then, $\widetilde{K} = \mathbf{Q}(\sqrt[4]{2}, i)$. Labelling the conjugates of $\alpha = \sqrt[4]{2}$ as

$$\alpha_1 = \sqrt[4]{2} \quad \alpha_2 = i\sqrt[4]{2} \quad \alpha_3 = -\sqrt[4]{2} \quad \alpha_4 = -i\sqrt[4]{2}$$

allows you to identify G with a subgroup of S_4. To identify D_4 with a subgroup of S_4, label the vertices of a square $v_1, \ldots, v_4$ in counterclockwise order. Then, $\rho = (1\ 2\ 3\ 4)$. The element r whose fixed field is K is then, under all these identifications, the permutation $(2\ 4)$, i.e. the reflection across the axis that goes through the vertices v_1 and

v_3. Note, for the fun of it, that r is then also complex conjugation. And for an awesome, though ultimately unnecessary, amount of fun, note that if you put $\alpha_1, \ldots, \alpha_4$ on the complex plane and connect the dots with four edges, you get a square and the D_4 acting on that square gives exactly the Galois action (compatibly with the above identifications). And the rotation ρ by $\pi/2$ is multiplication by i!)

(2) Chebotarev density applied to D_4-quartic fields. This questions continues with the setup of the previous question. There, your work allows you to determine the possible factorization types of primes of F in K. For each such factorization type for unramified primes of F, find the density of primes of F with that given factorization type in K/F.