Assignment 7 – All 4 parts – Math 661

Due in class: Thursday, Apr. 30, 2020

(1) Let $\zeta = \exp(2\pi i/7)$ and let $K = \mathbf{Q}(\zeta)$, the 7th cyclotomic field. It turns out that $u_1 := 1 + \zeta$ and $u_2 := 1 + \zeta^3$ form a fundamental system of units of K. The goal of this exercise is to show that the regulator of K is

$$R(K) = 4\left((\log|1+\zeta^3|)^2 - \log|1+\zeta| \cdot \log|1+\zeta^2|\right).$$

(a) Recall that the six complex embeddings $K \to \mathbf{C}$ are given by $\iota_j(\zeta) = \zeta^j$. Pair these up into $\tau_1, \overline{\tau}_1, \tau_2, \overline{\tau}_2, \tau_3, \overline{\tau}_3$. For $j = 1, 2$, let $U_j = (\tau_1(u_j), \tau_2(u_j), \tau_3(u_j))$. What are U_1 and U_2?

(b) Let $\widetilde{\mathrm{Log}}' : \mathcal{O}_K^\times \to H'$ be the map from Lecture 22, Part 3 (so H' is the "trace-zero" subspace of $\mathbf{R}^{r_1+r_2}$) and let $L_j := \widetilde{\mathrm{Log}}'(U_j)$. What are L_1 and L_2.

(c) Now, show that the regulator of K is as given above.

(2) Let F be a field and let $v : F \to \mathbf{R} \cup \{\infty\}$ be a valuation on F. We define the *valuation ring* of v by

$$\mathcal{O}_v := \{\alpha \in F : v(\alpha) \geq 0\}.$$

(a) Show that $\mathcal{O}_v$ is a subring of F.

(b) Show that $\mathcal{O}_v^\times = \{\alpha \in F : v(\alpha) = 0\}$.

(c) Conclude that the set $\mathfrak{m}_v := \{\alpha \in F : v(\alpha) > 0\}$ is a maximal ideal of $\mathcal{O}_v$. Further conclude that $\mathcal{O}_v$ is a local ring (i.e. that $\mathfrak{m}_v$ is the unique maximal ideal of $\mathcal{O}_v$).

(3) Let $\mathcal{O}$ be a Dedekind domain and let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}$. Recall that for I a non-zero fractional ideal of $\mathcal{O}$, we defined $\mathrm{ord}_\mathfrak{p}(I)$ to be the power of $\mathfrak{p}$ occurring in the unique factorization of I into prime ideals. Let F be the fraction field of $\mathcal{O}$. Define a function, also denoted $\mathrm{ord}_\mathfrak{p} : F \to \mathbf{R} \cup \{\infty\}$, by $\mathrm{ord}_\mathfrak{p}(\alpha) := \mathrm{ord}_\mathfrak{p}(\alpha\mathcal{O})$ for nonzero $\alpha \in F$ (and $\mathrm{ord}_\mathfrak{p}(0) := \infty$). Show that $\mathrm{ord}_\mathfrak{p}$ is a valuation on F.

(4) Let p be a prime and let q be another prime. On $\mathbf{Q}$, show that the p-adic absolute value $|\cdot|_p$ is not equivalent to $|\cdot|_q$, nor to the usual absolute value $|\cdot|$. (Hint: it's probably easiest here to use the criterion that $|\cdot|_1$ and $|\cdot|_2$ are equivalent if and only if for all $\alpha \in F$, $|\alpha|_1 < 1$ implies $|\alpha|_2 < 1$.)

(5) Valuation rings. One definition of a *valuation ring* is that it is an integral domain R such that for every element α of its fraction field F, $\alpha \in R$ or $\alpha^{-1} \in R$. We'll see that

each valuation ring is actually *the* valuation ring of some valuation on F, but in a more general sense than we previously defined. We'll say a few other things about valuation rings in this exercise. Valuation rings played a very central role in algebraic geometry in the mid-1900s as they offered an abstract way of talking about algebro-geometric spaces. They still play an important role in the subject.

(a) Suppose R is a valuation ring. Show that the set of ideals of R is totally ordered with respect to inclusion, and hence that R is a local ring.

(b) A more general definition of a valuation on a field goes as follows. A *totally ordered abelian group* is an abelian group Γ, typically denoted additively, equipped with a total order $\leq$ such that, for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$,

$$\gamma_1 \leq \gamma_2 \Rightarrow \gamma_1 + \gamma \leq \gamma_2 + \gamma.$$

Examples include $\mathbf{R}$ and $\mathbf{Z}$ with their usual ordering. A Γ-*valued valuation* on a field F is a function $v : F \to \Gamma \cup \{\infty\}$ such that

(i) $v(\alpha) = \infty$ if and only if $\alpha = 0$,

(ii) $v(\alpha\beta) = v(\alpha) + v(\beta)$, i.e. $v|_{F^\times}$ is a group homomorphism,

(iii) $v(\alpha + \beta) \geq \min(v(\alpha), v(\beta))$.

(so, we previously defined an $\mathbf{R}$-valued valuation; sometimes this more general notion here is called a *Krull valuation*). The image of $v|_{F^\times}$ is called the *value group* of v. We may still define *the* valuation ring of v to be

$$\mathcal{O}_v := \{\alpha \in F : v(\alpha) \geq 0\}$$

where 0 is the identity element of the group Γ. Show that the valuation ring of v is a valuation ring (as defined above), that

$$\mathfrak{p}_v := \{\alpha \in F : v(\alpha) > 0\}$$

is its maximal ideal, and that $\mathcal{O}_v^\times = \{\alpha \in F : v(\alpha) = 0\}$.

(c) Now, let's show that each valuation ring is *the* valuation ring of a valuation on its fraction field. First, let Γ be some abelian group and let Γ^+ be a submonoid of Γ (like $\mathbf{R}_{\geq 0}$ in $\mathbf{R}$). Define a relation $\leq$ on Γ by

$$\gamma_1 \leq \gamma_2 \text{ if } \gamma_2 - \gamma_1 \in \Gamma^+.$$

Show that $\leq$ is a preorder on Γ (i.e. it is reflexive and transitive, without necessarily being antisymmetric) and show that

$$\Gamma_0 := \{\gamma \in \Gamma : \gamma \geq 0 \text{ and } \gamma \leq 0\}$$

is a subgroup of Γ.

(d) Let $\overline{\Gamma} := \Gamma/\Gamma_0$ and for two cosets $\gamma_1 + \Gamma_0, \gamma_2 + \Gamma_0 \in \overline{\Gamma}$, define

$$\gamma_1 + \Gamma_0 \leq \gamma_2 + \Gamma_0 \text{ if } \gamma_1 \leq \gamma_2.$$

Suppose that for every element $\gamma \in \Gamma$, then $\gamma \in \Gamma^+$ or $-\gamma \in \Gamma^+$. Show that this is a well-defined *total* order that makes $\overline{\Gamma}$ a totally ordered abelian group.

(e) Finally, let R be a valuation ring with field of fractions F. Let $\Gamma = F^\times$ (which is an abelian group, though typically denoted multiplicatively) and let $\Gamma^+ = R\backslash\{0\}$. Show that the setup of the previous part with these choices turns $F^\times/R^\times$ into a totally ordered abelian group and that the map sending an element of $F^\times$ to its equivalence class in $F^\times/R^\times$ (and sending 0 to ∞) is a valuation whose valuation ring is R.

(6) Let $K = \mathbf{Q}(i)$.

(a) Let $p_\pm = 2 \pm i \in \mathcal{O}_K = \mathbf{Z}[i]$. Show that p_+ and p_- are prime in $\mathcal{O}_K$. Let $\mathfrak{p}_\pm = p_\pm \mathcal{O}_K$ and let $|\cdot|_\pm$ be the $\mathfrak{p}_\pm$-adic absolute value on K (i.e. the absolute value attached to the $\mathfrak{p}_+$-adic valuation $\mathrm{ord}_{\mathfrak{p}_+}$ (resp. $\mathfrak{p}_-$-adic valuation $\mathrm{ord}_{\mathfrak{p}_-}$) defined in Question 3). Show that $|\cdot|_+$ and $|\cdot|_-$ are inequivalent.

(b) Show that for all $a \in \mathbf{Q}$, $\mathrm{ord}_{\mathfrak{p}_+}(a) = \mathrm{ord}_{\mathfrak{p}_-}(a) = \mathrm{ord}_5(a)$, where ord_5 is the usual 5-adic valuation on $\mathbf{Q}$.

(c) Show that $p = 1 + i$ is prime in $\mathcal{O}_K$, let $\mathfrak{p} = (1+i)\mathcal{O}_K$. Show that for all $a \in \mathbf{Q}$, $\mathrm{ord}_2(a) = 2\,\mathrm{ord}_{\mathfrak{p}}(a)$, where ord_2 is the usual 2-adic valuation on $\mathbf{Q}$.

(7) Suppose F is a field equipped with a (nontrivial) non-archimedean absolute value $|\cdot|$.

(a) Show that a sequence a_n in F is a Cauchy sequence if and only if the sequence $a_{n+1} - a_n$ converges to 0.

(b) Suppose F is complete with respect to $|\cdot|$. Show that the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges if and only if a_n converges to 0. (In other words, unlike with the real numbers, the "nth term test" from calculus is both necessary and sufficient for the convergence

of the series. Calc 2 would be a lot easier if we just taught the students p-adically, right?!)