**Assignment 8 – All 2 parts – Math 661**

**Due in class: Thursday, May 7, 2020**

(1) DVR $\Leftrightarrow$ local Dedekind domain. In Lecture 28, we showed that a DVR is a local PID and I mentioned that the converse is true; that, in fact, you need merely assume that $R$ is a local Dedekind domain. In this exercise, you'll prove this.

(a) Suppose $R$ is a Dedekind domain with a finite number of prime ideals. Show that $R$ is a PID. (Hint: say $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are the non-zero prime ideals of $R$, it suffices to show that each $\mathfrak{p}_i$ is principal since every other ideal is a product of these. Use the Chinese Remainder Theorem to find an element $\alpha \in \mathfrak{p}_i \setminus \mathfrak{p}_i^2$ such that $\alpha \equiv 1 \pmod{\mathfrak{p}_k}$ for $k \neq i$ and show $\mathfrak{p}_i = (\alpha)$.)

(b) Suppose $R$ is a local Dedekind domain. Why does it only have finitely many prime ideals?

(c) From parts (a) and (b), we can conclude that a local Dedekind domain is a local PID. So, now it suffices to show that a local PID is a DVR. Let $R$ be a local PID with fraction field $F$ and let $t$ be a generator of its unique maximal ideal $\mathfrak{m}$. Show that for every non-zero $r \in R$, there is an $n_r \in \mathbf{Z}_{\geq 0}$ such that $r \in (t^{n_r}) \setminus (t^{n_r+1})$. Define $v : F \to \mathbf{R} \cup \{\infty\}$ by

$$v(r) = \begin{cases} n_r & r \neq 0, \\ \infty & r = 0. \end{cases}$$

Show that $v$ is a discrete valuation on $F$ and that $R$ is its valuation ring. Done!

(2) Let $K = \mathbf{Q}(i)$.

(a) Show that there are exactly two distinct embeddings $\iota$ and $\iota'$ of $K$ into $\mathbf{Q}_5$.

(b) Let $|\cdot|_5$ denote the usual 5-adic absolute value on $\mathbf{Q}_5$. For $\alpha \in \mathbf{Q}(i)$, define

$$|\alpha| := |\iota(\alpha)| \quad \text{and} \quad |\alpha|' := |\iota'(\alpha)|.$$

Determine $|2 \pm i|$ and $|2 \pm i|'$. (Compare this to Assignment 7, Question (6a).)

(3) Show that there are two square roots of 2 in $\mathbf{Q}_7$. Writing them as

$$\alpha = \sum_{n=0}^{\infty} a_n 7^n$$

and

$$\beta = \sum_{n=0}^{\infty} b_n 7^n$$

with $a_n, b_n \in \{0, 1, \ldots, 6\}$, find $a_n$ and $b_n$ for $n = 0, 1, 2, 3$.