Problem set 2 – Math 699 – Algebraic number theory

In order to get through all the material this semester with us only meeting once a week for lectures, you'll need to cover some material yourself. These exercises are meant to guide you some.

To begin with, there is some commutative algebra background that you should refresh/learn. Some canonical references for commutative algebra are Atiyah–Macdonald's *Introduction to commutaive algebra*, Eisenbud's *Commutative algebra with a view toward algebraic geometry*, and Matsumura's *Commutative ring theory*; though frankly, the material below is all presumably in any graduate algebra book such as Lang.

All rings are commutative with identity. Always.

(1) Let $\varphi : A \to B$ be a ring homomorphism and let $\mathfrak{P}$ be a prime ideal of B. Show that $\varphi^{-1}(\mathfrak{P})$ is a prime ideal of A. Show that if $\mathfrak{P}$ is maximal, then $\varphi^{-1}(\mathfrak{P})$ need not be maximal.

(2) Let A be a ring and recall that an A-module M is called *Noetherian* if every A-submodule of M is finitely-generated. The ring A is called *Noetherian* if it is Noetherian as a module over itself, i.e. if all of its ideals are finitely-generated.

(a) Recall that some people say that M is Noetherian if it satisfies the *ascending chain condition* (a.c.c.):

Every ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ of submodules of M stabilizes (i.e. eventually $M_n = M_{n+1} = M_{n+2} = \cdots$).

Show that the two definitions are equivalent. Show that they are both equivalent to: Every non-empty subset of submodules has a maximal element. (The equivalence between a.c.c. and this latter condition is just a result about posets).

(b) Noetherian property and factorization: Let A be a Noetherian ring and $\mathfrak{a}$ a non-zero ideal in A. Show that there are non-zero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ such that $\mathfrak{a} \supseteq \mathfrak{p}_1 \cdots \mathfrak{p}_r$. (Hint: suppose the set of all ideals of A not satisfying this condition is non-empty and obtain a contradiction).

(3) Basic facts about the Noetherian property: Prove them!

(a) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Then M is Noetherian if and only if both M' and M'' are Noetherian.

(b) If A is a Noetherian ring and M is a finitely-generated A-module, then M is Noetherian.

(c) Hilbert Basis Theorem: If A is Noetherian, then so is $A[x_1, \ldots, x_n]$. (This is is rather iconic result with a rather iconic proof; you need not prove this yourself, but it would be nice to look it up).

(d) Let $\varphi : A \to B$ be a surjective ring homomorphism and assume that A is Noetherian. Show that B is Noetherian.

(4) Let A be a domain and K its field of fractions. A *fractional ideal of A* is a non-zero A-submodule I of K such that there is an $a \in A$ with $aI \subseteq A$; to avoid confusion, on occasion one calls the usual ideals of A *integral ideals*. Let $a \in K$ be non-zero, by the *principal fractional ideal* (a), we mean aA. A fractional ideal I is called *invertible* if there is another fractional ideal, denoted I^{-1}, such that $II^{-1} = A$.

(a) Show that every principal fractional ideal is invertible.

(b) Show that every fractional ideal can be written as $I/J := IJ^{-1}$ where I and J are integral ideals.

(c) Show that any finitely-generated A-submodule of K is a fractional ideal.

(d) Conversely, if A is Noetherian, show that any fractional ideal of A is finitely generated.