Problem set 3 – Math 699 – Algebraic number theory

(1) Show that every UFD is integrally closed. (Hint: show that if f is a monic polynomial over a UFD A, then any root of f in the fraction field of A is actually in A)

(2) Let L/K be a finite separable extension of fields of degree n. Show that there are exactly n K-linear embeddings of L into an algebraic closure $\overline{K}$ of K. (Hint: write $L = K(\alpha)$ and let f be the minimal polynomial of α over K. Under a K-linear embedding of L into $\overline{K}$, α must go to a root of f.)

(3) Let L/K be a finite extension of fields of degree n. For $\alpha \in L$, let $m_\alpha : L \to L$ be the "multiplication by α" map $a \mapsto \alpha a$. Note that it is a linear transformation of the n-dimensional K-vector space L. Define the *trace of* α to be $\mathrm{tr}_{L/K}(\alpha) := \mathrm{tr}(m_\alpha)$, the *norm of* α to be $\mathcal{N}_{L/K}(\alpha) := \det(m_\alpha)$, and the *characteristic polynomial of* α to be $\mathrm{char}_\alpha(x) := \det(xI - m_\alpha) \in K[x]$.

   (a) Show that these are group homomorphisms $\mathrm{tr}_{L/K} : L \to K$ and $\mathcal{N}_{L/K} : L^\times \to K^\times$.

   (b) If L/K is separable, and $\sigma_1, \ldots, \sigma_n$ denote the n K-linear embeddings of L into $\overline{K}$, show that

$$\mathrm{tr}_{L/K}(\alpha) = \sum_{i=1}^{n} \sigma_i(\alpha), \qquad \mathcal{N}_{L/K}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha)$$

   and

$$\mathrm{char}_\alpha(x) = \prod_{i=1}^{n} (x - \sigma_i(\alpha)).$$

(4) Consider $\mathbf{R}^n$ with the standard inner product $\langle \ , \ \rangle$, and let $v_1, \ldots, v_n$ be n linearly independent (column) vectors in $\mathbf{R}^n$. Let A be the matrix whose columns are $v_1, \ldots, v_n$.

   (a) Explain why the volume of the parallelotope (higher-dimensional parallelogram) generated by the v_i is $|\det A|$.

   (b) Let B be the *Gram matrix* of the v_i, i.e. the matrix whose (i, j)-entry is $\langle v_i, v_j \rangle$. Show that the above volume is $\sqrt{|\det B|}$. (Hint: what is $A^{\mathrm{T}} A$?)

(5) Let $\alpha = a + bi$ and $\beta = c + di$ be two complex numbers and consider the rank 2 lattice $\langle \alpha, \beta \rangle \subseteq \mathbf{C}$ generated by them (ugh, the notation for a lattice generated by

two elements is the same as for the inner product of two elements. When did this happen?). Show that

$$\begin{vmatrix} a+bi & c+di \\ a-bi & c-di \end{vmatrix}^2.$$

is -4 times the square of the area of the fundamental parallelogram of this lattice. Hence, as far as measuring area of rank 2 lattices in $\mathbf{C}$, we can use the above invariant instead of the area.

(6) (Vandermonde determinant) Show that

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n} (\alpha_j - \alpha_i).$$