Problem set 8 – Math 699 – Algebraic number theory

(1) Let $K = \mathbf{Q}(\sqrt[3]{2})$ and take the power basis $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ for $\mathcal{O}_K$.

(a) Find the image of this basis under $j_{\mathbf{R}}$ in the Minkowski space $\mathbf{R}^3$.

(b) Show that this is an orthogonal basis of the lattice $j_{\mathbf{R}}(\mathcal{O}_K)$.

(2) Let K be a number field and let $\mathfrak{a}$ be an integral ideal of K. We define its *absolute norm* to be $\mathcal{N}(\mathfrak{a}) := [\mathcal{O}_K : \mathfrak{a}] = \#\mathcal{O}_K/\mathfrak{a}$. The use of the term 'norm' is because the ideal norm $\mathcal{N}_{K/\mathbf{Q}}(\mathfrak{a})$ is generated by the absolute norm of $\mathfrak{a}$.

(a) Show that if $\mathfrak{a} = \mathfrak{p}$ is prime and $\mathfrak{p}$ is a factor of the rational prime $p \in \mathbf{Z}$, then $\mathcal{N}(\mathfrak{a})$ is p^f for some $f \geq 1$. (And what do you think that f is?)

(b) Show that if $\mathfrak{p}$ and $\mathfrak{p}'$ are distinct primes of K, then $\mathcal{N}(\mathfrak{p}\mathfrak{p}') = \mathcal{N}(\mathfrak{p})\mathcal{N}(\mathfrak{p}')$ and $\mathcal{N}(\mathfrak{p}^e) = \mathcal{N}(\mathfrak{p})^e$. (Hint: Chinese Remainder Theorem and think of $\mathfrak{p}^i/\mathfrak{p}^{i+1}$ as an $\mathcal{O}_K/\mathfrak{p}$-vector space.) Conclude that the absolute norm is multiplicative.

(c) Show that if $\alpha \in \mathfrak{a}$, then $\mathcal{N}(\mathfrak{a}) \mid \mathcal{N}_{K/\mathbf{Q}}(\alpha)$ and $\mathcal{N}(\mathfrak{a})\mathbf{Z} = \mathcal{N}_{K/\mathbf{Q}}(\alpha)\mathbf{Z}$ if and only if $\mathfrak{a} = \alpha\mathcal{O}_K$.

(3) The Minkowski bound can be used to show that certain ideals are principal. Recall that for a number field K, the *Minkowski bound* is $M_K := \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n}\sqrt{|\Delta(K)|}$ (well, I think in class I called this B_K...). Show that if $M_K < 2$, then every ideal of K is principal, i.e. K has class number 1. (By my count, there are 16 such fields.)

(4) For the following number fields K and sets X of ideals of K, show that every ideal in X is principal.

(a) $K = \mathbf{Q}(\sqrt{17})$, $X =$ the primes dividing 2. (Hint: first, recall that we had a theorem that gives a set of two generators for the primes dividing a rational prime, see e.g. Neukirch Prop. 8.3; second, here, show that one of the ideals is generated by $(3 - \sqrt{17})/2$ while the other is generated by $(-5 + \sqrt{17})/2$).

(b) $K = \mathbf{Q}(\sqrt[3]{2})$, $X =$ the prime dividing 3.

(c) $K = \mathbf{Q}(\sqrt{2}, \sqrt{3})$, $X =$ the prime dividing 2. (Hint: your work Question (7) of Problem Set 4 might be useful).

(5) Show that the primes dividing 2 and 3 in $K = \mathbf{Q}(\sqrt{-5})$ are *not* principal.