# THE LENGTH OF DUCCI'S FOUR-NUMBER GAME

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ABSTRACT. The length of a Ducci 4-number game, defined immediately below, is at most six if the initial vector is not cyclically monotone. If it is cyclically monotone, then the length is shown here to be, with error at most 5.4, a linear function of the logarithm of the Euclidean distance from the initial vector (normalized) to the unique normalized vector of infinite length.

**1. The main theorem.** Define $T : \mathbf{R}^4 \to \mathbf{R}^4$ by the formula

$$T(a, b, c, d) = (|b - a|, |c - b|, |d - c|, |d - a|).$$

For each $v \in \mathbf{R}^4$ the sequence $(T^n(v))_{n \geq 0}$ is called a *Ducci sequence*, or *4-number game*. A minimal $n \geq 0$ with $T^n(v) = 0$ is called the *length* of $v$, or of the 4-number game; if there is no such $n$, then $v$ is said to have *infinite length*. Lengths of 4-number games, and more generally of $n$-number games, have been one of the major topics in the study of Ducci sequences. (See any of the references below, but especially [**1, 4–6**].) In this paper we give a simple method for estimating with error less than 5.4 the length of any 4-number game.

As in [**6**], a vector $v \in \mathbf{R}^4$ is said to be *equivalent* to $w \in \mathbf{R}^4$ if $w$ is in the orbit of $v$ under the action on $\mathbf{R}^4$ of the group generated by the permutations $R$, $S$, $A_f$ (for $f \in \mathbf{R}$) and $M_e$ (for $0 \neq e \in \mathbf{R}$) of $\mathbf{R}^4$ defined by the formulas $R(a, b, c, d) = (d, c, b, a)$, $S(a, b, c, d) = (d, a, b, c)$, $M_e(v) = ev$, and $A_f(v) = v + (f, f, f, f)$. We also call $v$ *normalized* if it has the form $(0, x, y, 1)$ where $0 < x < y \leq 1 - x$.

In Section 3 we will see that the problem of estimating length reduces to the case of normalized vectors. (Briefly, if a vector is cyclically monotone, then its length equals that of an equivalent normalized vector, and otherwise the length is at most 6 and hence is known to within 5.4.) We now state our main theorem and give a simple estimate

for the length of a normalized vector. This estimate will be illustrated by examples in Section 2. In Sections 4 and 5 we will prove the main theorem and show that the estimate has error less than 5.4.

In the theorem below and throughout the rest of the paper, we let

$$r = \left( 2 - \frac{2\sqrt{33}}{9} \right)^{1/3} + \left( 2 + \frac{2\sqrt{33}}{9} \right)^{1/3} \approx 2.38$$

denote the unique real zero of $x^3 - 4x - 4$ and also set

$$\mu = \frac{2}{\ln(r+1)} \approx 1.64$$

and

$$v_\infty = \left( 0, \frac{6 - r^2}{2}, \frac{r^2 - 2r}{2}, 1 \right) \approx (0, .16, .46, 1).$$

**1.1 Main theorem.** *There exist constants $C$ and $D$ such that if $d$ is the Euclidean distance from a normalized vector $v$ to $v_\infty$, where $v \neq v_\infty$, then the length $L$ of $v$ satisfies*

$$C < L - \mu \, \ln \frac{1}{d} < D.$$

**1.2** *Remark: The simple estimate.* The proof of the main theorem will show that one can choose $C = -2.4$ and $D = 8.4$, so that

$$\left| L - \left( 3 + \mu \, \ln \frac{1}{d} \right) \right| < 5.4.$$

We will refer to $3 + \mu \, \ln 1/d$ as the *simple estimate* for $L$. As a first example, the vector $(0, \sqrt{2}, e + \sqrt{3}, \pi^2)$ has length 10; the simple estimate of its length (rounded) is 9.6. More examples appear in Section 2 including one of a vector in $\mathbf{Z}^4$ of length 3782 (the simple estimate is about 3782.1). The lower bound $-2.4 + \mu \, \ln 1/d$ for $L$ is of interest only if $d < e^{-7.4/\mu} \approx 0.011$, since otherwise $-2.4 + \mu \, \ln 1/d \leq 5$ and, by Lemma 3.3 below, normalized vectors always have length at

least 5. The above values for $C$ and $D$ can be improved by more careful numerical work, see Remark 5.2.

Note that Theorem 1.1 implies that, if $v \in \mathbf{Z}^4$, then $v$ has finite length. This observation is attributed to Professor Ducci in [2] and the observation is generalized to arbitrary $2^n$-number games and proven. Theorem 1.1 of course implies that all vectors $v \in \mathbf{R}^4$ have finite length except for those equivalent to $v_\infty$. The fact that up to equivalence there is exactly one element of $\mathbf{R}^4$ of infinite length was first proved by Lotan [3]. One can give a rather short proof of this fact using ideas in this paper but without using the sledgehammer of the main theorem. Such a proof is sketched in an appendix.

**2. Examples.** In this section we give some examples based on the "simple estimates" of Remark 1.2.

2.1 *Example*: *Tribonacci numbers.* The Tribonacci numbers $t_0, t_1, t_2,$ ... are defined by the equations $t_0 = 0$, $t_1 = t_2 = 1$ and for $n \geq 3$, $t_n = t_{n-1} + t_{n-2} + t_{n-3}$. Webb [7] has shown that the "Tribonacci vector" $T_n = (t_n, t_{n-1}, t_{n-2}, t_{n-3})$ has length $3\,[n/2]$ for all $n \geq 3$ and that any vector in $\mathbf{Z}^4$ whose coordinates all are between 0 and $t_n$ (inclusive) has length at most $3\,[n/2] + 1$. Each $T_n$ is equivalent to the normalized vector

$$\left( 0, \ \frac{t_{n-2} - t_{n-3}}{t_n - t_{n-3}}, \ \frac{t_{n-1} - t_{n-3}}{t_n - t_{n-3}}, \ 1 \right).$$

The following tables record the length of $T_n$ for all $n$ from 5 to 24, together with the simple estimate of its length, rounded to the nearest tenth.

| n | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|----|----|----|----|----|
| length | 6 | 9 | 9 | 12 | 12 | 15 | 15 | 18 | 18 | 21 |
| estimate | 8.1 | 9.3 | 10.2 | 12.3 | 14.1 | 14.7 | 16.5 | 18.9 | 19.4 | 20.8 |

| $n$ | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|----|----|----|----|----|----|----|----|----|----|
| length | 21 | 24 | 24 | 27 | 27 | 30 | 30 | 33 | 33 | 36 |
| estimate | 23.3 | 24.1 | 25.2 | 27.6 | 28.8 | 29.7 | 31.8 | 33.6 | 34.3 | 36.0 |

2.2 *Example*: *Decimal approximations*. Here are lengths and estimated lengths for the vectors $(0, x_n, y_n, 1)$ where $x_n$ and $y_n$ are obtained by truncating to $n$ digits the decimal expansions of the second and third coordinates of $v_\infty$, respectively. Thus, for example, $x_2 = 0.16$ and $y_2 = 0.45$.

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 100 | 1000 |
|---|---|---|---|---|---|---|---|---|---|
| length | 22 | 24 | 27 | 29 | 34 | 38 | 41 | 382 | 3782 |
| estimate | 21.0 | 23.7 | 25.6 | 29.5 | 33.5 | 37.3 | 41.0 | 381.5 | 3782.1 |

If a normalized vector has small length, then the simple estimate for its length is both unnecessary and uninformative. The main theorem shows that the normalized vectors of large length are those close to $v_\infty$. The previous example focused on one way of coming up with vectors close to $v_\infty$; the next gives a more natural way to do this.

2.3 *Example*: *Convergents*. In the $m$th row and $n$th column of the table below, we have listed the length and the simple estimate of length, respectively, of the vector $(0, x_m, y_n, 1)$, where $x_m$ and $y_n$ are the $m$th and $n$th convergents to the continued fraction expansions of the second and third coordinates of $v_\infty$, respectively. Because convergents give good approximations relative to the size of their denominators, by multiplying $(0, x_m, y_n, 1)$ by the product of the denominators of $x_m$ and $y_n$, we can hope to get elements of $\mathbf{Z}^4$ of large length relative to the magnitude of their coordinates.

|   | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 6, 8.1 | 11, 11.3 | 11, 11.4 | 11, 11.4 | 11, 11.4 | 11, 11.4 |
| 2 | 6, 8.1 | 13, 12.3 | 15, 14.8 | 15, 14.9 | 15, 14.9 | 15, 14.9 |
| 3 | 6, 8.1 | 13, 13.4 | 17, 16.9 | 25, 25.5 | 25, 25.6 | 25, 25.6 |
| 4 | 6, 8.1 | 13, 13.4 | 17, 16.9 | 27, 27.6 | 35, 35.1 | 34, 35.2 |
| 5 | 6, 8.1 | 13, 13.4 | 17, 16.9 | 27, 27.6 | 37, 37.1 | 39, 38.6 |
| 6 | 6, 8.1 | 13, 13.4 | 17, 16.9 | 27, 27.6 | 37, 37.2 | 41, 40.7 |

One might observe that the fourth row of the above table gives an example of a pair of vectors for which the vector *closer* to $v_\infty$ actually has the smaller length. Note that

$$(0, x_3, y_4, 1) = \left( 0, \frac{9}{56}, \frac{47}{103}, 1 \right)$$

is equivalent to

$$(0, 927, 2632, 5768) = (0, t_{13}, t_{13} + t_{14}, t_{16}),$$

which is essentially the vector (in the case $n = 16$) that Webb constructs [**7**, last displayed formula, page 35] to give a vector of maximal length among those vectors in $\mathbf{Z}^4$ with largest coordinate at most $t_{16}$. It might be interesting to see if there are other connections between the examples arising from Tribonacci numbers and from convergents.

2.4 *Last example.* The largest discrepancy we have observed between actual length and the simple estimate of length occurs with vectors of the form $(0, \varepsilon, (1/3) + \varepsilon, 1)$ where $\varepsilon$ is chosen very small and positive. Then the lengths are all eight but the estimated lengths are about 5.6. That the simple estimate predicts length as well as it does in the above examples suggests that tighter bounds are possible for length and, further, that the upper and lower bounds for length that we have given are, essentially by accident, about equally far from these hypothesized tighter upper and lower bounds.

**3. Normalization.** In this section, for which we make no great claim to originality, we prove some basic lemmas, including for the sake of completeness some proofs of known results. We use the maps $R$, $S$, $A_f$ and $M_e$ from Section 1.

Let $v \in \mathbf{R}^4$.

3.1 *Definition.* We say $v$ is *cyclically monotone* if for some $i \geq 0$ we have $S^i v = (e, f, g, h)$ where $e < f < g < h$ or $e > f > g > h$.

The reduction of the problem of estimating lengths to the normalized case is accomplished by the next two lemmas.

**3.2 Lemma.** *If $v$ is cyclically monotone, then it is equivalent to a normalized vector of $\mathbf{R}^4$, and any normalized vector equivalent to $v$ has the same length as $v$.*

In Proposition 6.1 below we will show that a cyclically monotone vector is equivalent to only *one* normalized vector. The proof of Lemma 3.2 indicates at least implicitly how to construct this normalized vector.

*Proof.* By the definition of equivalence we may assume without loss of generality, i.e., after suitable applications of $R$ and $S$, that $v = (a, b, c, d)$ where $a < b < c < d$. Applying $M_{1/(d-a)} A_{-a}$ we may assume $v$ has the form $(0, b, c, 1)$. If $b+c \leq 1$ we are finished, so suppose $b + c > 1$. Then

$$RA_1 M_{-1}(v) = (0, 1 - c, 1 - b, 1),$$

which is clearly normalized, so there is a normalized vector equivalent to $v$. That equivalent cyclically monotone vectors have the same length follows from the fact that the bijections $R$, $S$, $M_e$ and $A_f$ clearly preserve length when applied to nonconstant vectors, i.e., to vectors not of the form $(g, g, g, g)$.          $\square$

The cyclically monotone vector $(0, 1, 2, 3)$ has length 5 and the vector $(0, 2, 5, 1)$ has length 6. The next lemma gives a sense in which these lengths are extremal.

**3.3 Lemma.** *If $v$ is not cyclically monotone, then its length is at most $6$. If it is cyclically monotone, then its length is at least $5$.*

*Proof.* The first assertion of the lemma is a standard result [**5**, Property F] and can be easily proved by considering in the order given below the cases of vectors in $\mathbf{R}^4$ of the following forms: $(a, b, a, b)$; $(a, b, a, c)$; $(a, m, b, M)$ where $M \geq a \geq m$, $M \geq b \geq m$; and, $(M, a, b, m)$ where $m \leq a \leq b \leq M$. Any vector which is not cyclically monotone is equivalent to a vector in one of the last two forms.

Now suppose $v$ is cyclically monotone; by the previous lemma we may suppose that $v$ is a normalized vector $(0, x, y, 1)$. Note that

$1-x \geq y \geq 2x-y$, and $1-x > y-2x$, so that in any case $1-x \geq |2x-y|$. It follows easily that the third coordinate of $T^4(v)$ is the absolute value of $y - |2x - y|$, which is clearly nonzero. Thus the length of $v$ is at least 5.   $\square$

We end this section with two lemmas in the spirit of the two above. We say a vector $(a, b, c, d) \in \mathbf{R}^4$ is *monotone* if $0 \leq a < b < c < d$ and *subnormal* if it is monotone and in addition $b + c \leq a + d$. Thus normalized vectors are subnormal; subnormal vectors are monotone; and monotone vectors are cyclically monotone.

**3.4 Lemma.**  *If $v = (a, b, c, d)$ is subnormal and $T(v)$ is cyclically monotone, then $T(v)$ is subnormal.*

*Proof.*  Since $v$ is monotone, $T(v) = (b - a, c - b, d - c, d - a)$. If $T(v)$ is not monotone, then since $d - a$ is clearly its largest coordinate, we therefore must have $b - a > c - b > d - c$, which contradicts the hypothesis that $b + c \leq a + d$. Also, because $b > a$, we have $(c - b) + (d - c) \leq (b - a) + (d - a)$, so $T(v)$ is indeed subnormal.   $\square$

**3.5 Lemma.**  *Define $U : \mathbf{R}^4 \to \mathbf{R}^4$ by the formula $U(a, b, c, d) = (b - a, c - b, d - c, d - a)$. If $v$ is subnormal and $U^n(v)$ is not monotone for some $n \geq 1$, then $T^{n+6}(v) = 0$.*

*Proof.*   Note that $T(v) = U(v)$ whenever $v$ is monotone.  By Lemma 3.3 it suffices to show that $T^k(v)$ is not cyclically monotone for some $k \leq n$. Suppose otherwise. Then by the previous lemma $T^k(v)$ is monotone for all $k \leq n$, and so $U^n(v) = T^n(v)$. This contradicts the hypothesis that $U^n(v)$ is not monotone.   $\square$

**4. Main lemma.**  In this section we give a geometric interpretation of the point $v_\infty$; this leads to our main lemma 4.5, which shows how bounds on the distances from $((6 - r^2)/2, (r^2 - 2r)/2)$ to various geometric objects lead to bounds on the lengths of normalized vectors. Related geometric methods are used in [**6**] to study the distribution of lengths.

4.1 *Notation.* Define the linear map $U : \mathbf{R}^4 \to \mathbf{R}^4$, cf. Lemma 3.5, by the rule

$$U(a, b, c, d) = (b - a, c - b, d - c, d - a).$$

Let $v = (0, x, y, 1) \in \mathbf{R}^4$ and for each $n \geq 0$ write

$$U^n(v) = (a_n, b_n, c_n, d_n).$$

By inspection,

$$a_0 = 0, \quad a_1 = x, \quad a_2 = y - 2x, \quad a_3 = 3x - 3y + 1.$$

Also for each $n \geq 0$,

$$(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) = (b_n - a_n, c_n - b_n, d_n - c_n, d_n - a_n)$$

so

$$b_n = a_n + a_{n+1},$$
$$c_n = b_n + b_{n+1} = a_n + 2a_{n+1} + a_{n+2},$$

and

$$d_n = c_n + c_{n+1} = a_n + 3a_{n+1} + 3a_{n+2} + a_{n+3}.$$

But also $d_{n+1} = d_n - a_n$, and hence,

$$a_{n+4} + 3a_{n+3} + 3a_{n+2} + a_{n+1} = 3a_{n+1} + 3a_{n+2} + a_{n+3},$$

so the sequence $(a_n)_{n \geq 0}$ satisfies what we will call the *first recursion relation*

$$(1) \qquad\qquad X_{n+4} = 2(X_{n+1} - X_{n+3}).$$

We now define a sequence of linear polynomials $A_n = A_n(X, Y) \in \mathbf{R}[X, Y]$ by setting

$$A_0 = 0, \quad A_1 = X, \quad A_2 = Y - 2X, \quad A_3 = 3X - 3Y + 1$$

and requiring the sequence $(A_n)_{n \geq 0}$ to satisfy the first recursion relation (1) for all $n \geq 0$. Thus,

$$A_n(x, y) = a_n \quad \text{for all} \quad n \geq 0.$$

Note that $A_4 = 2(A_1 - A_3) = 6Y - 4X - 2$.

4.2 *Notation.* We let $\bar{z}$ and $\text{Tr}\,(z) = z + \bar{z}$ denote the complex conjugate and trace of a complex number $z$, respectively. Recall the real zero $r$ of $x^3 - 4x - 4$; write

(2) $$x^3 - 4x - 4 = (x - r)(x - s)(x - \bar{s}),$$

where $s$ denotes the nonreal zero whose coefficient of $i$ is positive, so $s \approx -1.19 + 0.51\,i$. We also let

$$\alpha = \frac{9r^2 - 8r - 24}{44} \quad \text{and} \quad \beta = \frac{9s^2 - 8s - 24}{44},$$

so $\alpha \approx 0.18$ and $\beta \approx -0.09 - 0.34\,i$. For all $n \geq 0$, let

$$M_n = \alpha r^n + \beta s^n + \bar{\beta}\,\bar{s}^n = \alpha r^n + \text{Tr}\,(\beta s^n),$$

and let $N_n = (1/2)M_{n+2}$ and

$$D_n = N_n + M_n + M_{n+1} = M_n + M_{n+1} + \frac{1}{2}M_{n+2}.$$

We also set $P_n = (M_n/D_n, N_n/D_n)$. In the next lemma we examine the values of the polynomials $A_m$ on the points $P_n$.

**4.3 Lemma.** *For all $n \geq 0$, $A_{n+1}(P_n) = A_{n+2}(P_n) = A_n(P_{n+1}) = 0$. Also, for all $n > 0$, $A_n(P_{n+j}) > 0$ whenever $j = 0$ or $j \geq 2$. Finally, for all $n \geq 0$, $A_{n+3}(P_n) > 0$.*

The proof of Lemma 4.3 will develop a useful recursion relation satisfied by the $M_n$, $N_n$ and $D_n$ sequences, and will also give a table (which will be needed below) of values of $M_n$, $N_n$, $D_n$ and $P_n$ for small $n$. Together these will imply that for all $n$ the numbers $M_n$, $N_n$ and $D_n$ are nonnegative integers with $D_n > 0$.

*Proof.* Note that the last fact of the lemma is implied by the earlier ones, since by the first recursion relation (1), if $n \geq 1$, then

$$A_{n+3}(P_n) = 2(A_n(P_n) - A_{n+2}(P_n)) > 0,$$

and also $A_3(P_0) = 1 > 0$.

Since by definition $r^3 = 4r + 4$ and similarly for $s$ and $\bar{s}$, it follows that the sequence $(M_n)_{n \geq 0}$ satisfies the recursion relation

$$(3) \qquad\qquad X_{n+3} = 4X_{n+1} + 4X_n$$

for all $n \geq 0$. We call (3) the *second recursion relation*. It is of course also satisfied by the sequences $(N_n)_{n \geq 0}$ and $(D_n)_{n \geq 0}$. By equation (2) we have $r + s + \bar{s} = 0$, $rs\bar{s} = 4$, $rs + r\bar{s} + s\bar{s} = -4$, and hence

$$r^2 + s^2 + \bar{s}^2 = (r + s + \bar{s})^2 - 2(rs + r\bar{s} + s\bar{s}) = 8.$$

Using these identities, the definitions of $M_n$, $N_n$ and $D_n$, and their recursion relation (3), one easily verifies the values in the following table.

| $n$ | 0 | 1 | 2 | 3 | 4 |
|------|-------|-----------------|------------|------------------|------------------|
| $M_n$ | 0 | 1 | 0 | 4 | 4 |
| $N_n$ | 0 | 2 | 2 | 8 | 16 |
| $D_n$ | 1 | 3 | 6 | 16 | 36 |
| $P_n$ | $(0,0)$ | $((1/3),(2/3))$ | $(0,(1/3))$ | $((1/4),(1/2))$ | $((1/9),(4/9))$ |

For each pair $m$, $n$ of nonnegative integers we set $K_{m,n} = D_n A_m(P_n)$.

Fix $m$ for the moment and write $A_m = aX + bY + c$. Then

$$K_{m,n} = aM_n + bN_n + cD_n.$$

Therefore, the sequence $(K_{m,n})_{n \geq 0}$ satisfies the second recursion relation (3) since the sequences $(M_n)$, $(N_n)$ and $(D_n)$ all satisfy it.

Also, for fixed $n$ the sequence $(K_{m,n})_{m \geq 0}$ satisfies the first recursion relation (1) since the polynomials $A_m$ satisfy it.

**Claim.** *For all $n \geq 1$, $K_{n,n} = 2^{n-1}$ and $K_{n,n\pm1} = 0$.*

*Proof of Claim.* Using the values of $D_n$, $A_n$ and $P_n$ computed explicitly above, it is routine to verify the claim for all $n \leq 3$. Now suppose inductively that $n > 3$ and that the claim is valid for all smaller integers. Using this hypothesis and the second recursion relation (3) on sequences of the form $(K_{m,n})_{n\geq0}$, we deduce in turn that

$$K_{n-3,n-1} = 4(K_{n-3,n-3} + K_{n-3,n-4}) = 4(2^{n-4} + 0) = 2^{n-2}$$

and similarly that $K_{n-3,n} = 2^{n-2}$, $K_{n-3,n+1} = 2^n$, $K_{n-2,n} = 2^{n-1}$, $K_{n-2,n+1} = 2^{n-1}$, and $K_{n-1,n+1} = 2^n$. Now using the fact that sequences of the form $(K_{m,n})_{m\geq0}$ satisfy the first recursion relation, (1) we deduce that

$$K_{n,n-1} = 2(K_{n-3,n-1} - K_{n-1,n-1}) = 2(2^{n-2} - 2^{n-2}) = 0$$

and, similarly, that $K_{n,n} = 2^{n-1}$ and $K_{n,n+1} = 0$. The claim is proved.
□

We can now verify the assertions of the lemma. Without loss of generality, we may assume that $n \geq 1$ (since $A_0 = 0$). Then by the claim $A_n(P_n) = K_{n,n}/D_n > 0$ and $A_{n+1}(P_n) = K_{n+1,n}/D_n = 0$. Similarly, $A_n(P_{n+1}) = K_{n,n+1}/D_{n+1} = 0$. If $n = 1$, then one checks that $A_{n+2}(P_n) = 0$; if $n > 1$, then the first recursion relation shows that

$$A_{n+2}(P_n) = 2(A_{n-1}(P_n) - A_{n+1}(P_n)) = 0.$$

Finally, an easy induction argument using the above claim and the fact that the sequences $(D_n)_{n\geq0}$ and $(K_{m,n})_{n\geq0}$ satisfy the second recursion relation establishes that for $j \geq 2$, $A_n(P_{n+j}) = K_{n,n+j}/D_{n+j} > 0$.
□

Let $P_\infty = ((6 - r^2)/2, (r^2 - 2r)/2)$, so that the coordinates of $P_\infty$ are just the middle coordinates of $v_\infty$.

**4.4 Lemma.**
$$P_\infty = \lim_{n\to\infty} P_n.$$

*Proof.* First,

$$
\begin{aligned}
\frac{M_n}{D_n} &= \frac{M_n}{M_n + M_{n+1} + (1/2)M_{n+2}} \\
&= \frac{\alpha r^n + \mathrm{Tr}\,(\beta s^n)}{\alpha(1 + r + r^2/2)r^n + \mathrm{Tr}\,((1 + s + s^2/2)s^n\beta)} \\
&= \frac{\alpha + \mathrm{Tr}\,((s/r)^n\beta)}{\alpha(1 + r + r^2/2) + \mathrm{Tr}\,((s/r)^n(1 + s + s^2/2)\beta)} \\
&\longrightarrow \frac{1}{1 + r + r^2/2} = \frac{6 - r^2}{2}
\end{aligned}
$$

as $n \to \infty$ since $|s/r| < 1$. Similarly,

$$
\begin{aligned}
\frac{N_n}{D_n} &= \frac{1}{2}\frac{M_{n+2}}{D_n} \\
&= \frac{\alpha r^{n+2} + \mathrm{Tr}\,(\beta s^{n+2})}{\alpha(2 + 2r + r^2)r^n + \mathrm{Tr}\,(\beta(2 + 2s + s^2)s^n)} \\
&\longrightarrow \frac{r^2}{r^2 + 2r + 2} = \frac{2}{r + 2} = \frac{r^2 - 2r}{2}.
\end{aligned}
$$

This completes the proof that $P_n \to P_\infty$ as $n \to \infty$.     □

For any line $L$ in $\mathbf{R}^2$ and any points $P$, $Q$ in $\mathbf{R}^2$, we let $\mathrm{dist}\,(P, Q)$ and $\mathrm{dist}\,(P, L)$ denote the Euclidean distance from $P$ to $Q$ and to $L$, respectively. We also write

$$
\omega = (r + 1)^{-1/2} = \left|\frac{s}{r}\right| \approx 0.544.
$$

Finally, $L_n$ will denote the line whose equation is $A_n = 0$.

**4.5 Main Lemma.** *Suppose $B$ and $G$ are positive numbers such that for all $n \geq 0$*

$$
\mathrm{dist}\,(P_n, P_\infty) \leq B\omega^n \quad and \quad B \geq \omega\,\mathrm{dist}\,(P_\infty, (1, 0)),
$$

*and for all $n \geq 1$,*
$$
\mathrm{dist}\,(P_\infty, L_n) \geq G\omega^n.
$$

*Then the length $L$ of any normalized vector $v = (0, x, y, 1) \neq v_\infty$ satisfies*

$$\frac{\ln\ d}{\ln\ \omega} - \frac{\ln\ G}{\ln\ \omega} + 1 \leq L < \frac{\ln\ d}{\ln\ \omega} - \frac{\ln\ B}{\ln\ \omega} + 9,$$

*where $d = \operatorname{dist}((x, y), P_\infty)$.*

*Proof.* We will use the notation of Notation 4.1 so that for all $n \geq 0$ we write $a_n = A_n(x, y)$ and $U^n(v)$ equals

$$(a_n, a_n + a_{n+1}, a_n + 2a_{n+1} + a_{n+2}, a_n + 3a_{n+1} + 3a_{n+2} + a_{n+3}).$$

Since $v$ is normalized, $(x, y)$ lies in the triangular region bounded by $x = 0$, $y = 0$, and $x + y = 1$. The point of this triangular region farthest from $P_\infty$ is $(1, 0)$, so

$$B \geq \omega \operatorname{dist}(P_\infty, (1, 0)) > \omega\, d,$$

where the first inequality is a hypothesis of the lemma. Since $\omega < 1$, there exists a least $n \geq 0$ with $d \geq B\omega^n$. If $n > 0$, then of course $d < B\omega^{n-1}$; this is also true if $n = 0$ by the remarks above.

If $k \geq 0$, then

$$\operatorname{dist}(P_\infty, P_{n+k}) \leq B\omega^{n+k} \leq B\omega^n \leq d.$$

Hence $(x, y)$ is not inside the triangle $P_n P_{n+1} P_{n+2}$. (The closed disk with center $P_\infty$ and radius $d = \operatorname{dist}(P_\infty, (x, y))$ contains this triangle.) The sides of this triangle are the lines $L_{n+1}$, $L_{n+2}$, $L_{n+3}$ by Lemma 4.3. Thus, for some $k \in \{1, 2, 3\}$, the point $(x, y)$ cannot be on the same side of $L_{n+k}$ as $P_{n+k}$ (if $k \neq 3$) or $P_n$ (if $k = 3$). (The interior of the triangle is the intersection of the open half planes $A_{n+j} > 0$ for $j = 1, 2, 3$.) Hence, $A_{n+k}(x, y) \leq 0$. Thus, there is a smallest positive $t \leq n + 3$ with $A_t(x, y) \leq 0$. Therefore, the first coordinate of $U^t(v)$ is not positive, so $U^{t-1}(v)$ cannot be monotone. Thus, there exists a least $j \leq t - 1 \leq n + 2$ with $U^j(v)$ not monotone. Hence by Lemma 3.5, $(0, x, y, 1)$ has length at most $j + 6 \leq n + 8$. Now, by the choice of $n$,

$$\ln\ B + (n - 1)\ln\ \omega > \ln\ d$$

so, keeping in mind that $\ln \omega < 0$,

$$L \leq 8 + n < 8 + 1 + \frac{\ln d}{\ln \omega} - \frac{\ln B}{\ln \omega},$$

proving the upper bound for $L$ in the main lemma.

We may assume that there exists a unique integer $n \geq 1$ with

$$(4) \qquad G\ \omega^{n+1} \leq d < G\ \omega^n.$$

(The use of the symbol $n$ here has no connection with its use in the previous paragraph.) After all, if this is not true, then $d \geq G\omega$ so that

$$1 + \frac{\ln d}{\ln \omega} - \frac{\ln G}{\ln \omega} \leq 2 \leq L$$

(Lemma 3.3), proving the lemma in this case. But then (4) implies that

$$(5) \qquad \text{dist}\ (P_\infty, (x, y)) < G\omega^n \leq \text{dist}\ (P_\infty, L_n)$$

so $(x, y)$ lies on the same side of $L_n$ as $P_\infty$. Now, for large $N$, $A_n(P_N) > 0$, so by Lemma 4.4 we have $A_n(P_\infty) \geq 0$. But (5) then implies that $A_n(P_\infty) > 0$, so $A_n(x, y) > 0$. Similarly, if $1 \leq k \leq n$, then

$$\text{dist}\ (P_\infty, (x, y)) < G\omega^n \leq G\omega^k \leq \text{dist}\ (P_\infty, L_k),$$

so $a_k = A_k(x, y) > 0$. Therefore, if $1 \leq k \leq n - 2$, then

$$U^k(v) = (a_k, a_k + a_{k+1}, a_k + 2a_{k+1} + a_{k+2}, a_k + 3a_{k+1} + 3a_{k+2} + a_{k+3})$$

has its first three terms positive. Hence, for all $k \leq n - 3$, $U^k(v)$ is monotone and hence $T^k(v) = U^k(v)$. Thus, by Lemma 3.3, $T^{n-3}(v)$ has length at least 5, so $v$ has length at least $(n-3) + 5 = n + 2$. But, from (4) we have $\ln G / \ln \omega + n + 1 \geq \ln d / \ln \omega$, so

$$L \geq n + 2 \geq 1 + \frac{\ln d}{\ln \omega} - \frac{\ln G}{\ln \omega}. \qquad \square$$

**5. Proof of the main theorem.** In this section we will find numbers $B$ and $G$ satisfying the conditions of the main lemma 4.5,

thereby proving the main theorem 1.1. The bounds found will be sharp enough to imply the assertion of Remark 1.2 that the simple estimate of length differs from true length by less than 5.4. In particular, we will show that for all $n \geq 0$,

$$\text{(6)} \qquad\qquad \text{dist}\,(P_n, P_\infty) < 0.685\,\omega^n$$

and that for all $n \geq 1$ we have

$$\text{(7)} \qquad\qquad \text{dist}\,(P_\infty, L_n) > 0.135\,\omega^n.$$

The inequality (6) is easily checked when $n \leq 2$ using the values of $P_n$ in the table in the proof of Lemma 4.3. (The value 0.685 was obtained by rounding upward the exact value of $\text{dist}\,(P_2, P_\infty)/\omega^2$, namely, $\sqrt{7r^2 - 4r - 26}/3$.) Now suppose $n \geq 3$. Then $\omega^3 \geq \omega^n$ and $1.116 r^n < D_n$, as is easily checked using the fact that both the sequences $(D_n)$ and $(r^n)$ satisfy the second recursion relation. It is convenient to write

$$\delta_r = r^2/2 + r + 1 \quad \text{and} \quad \delta_s = s^2/2 + s + 1,$$

so that $P_\infty = (1/\delta_r, r^2/2\delta_r)$, as was noted in the proof of Lemma 4.4.

The triangle inequality implies the useful formula

$$(\text{Tr}\,(z))^2 + (\text{Tr}\,(w))^2 \leq 2\big(|z|^2 + |w|^2 + |z^2 + w^2|\big).$$

Combining these observations we can calculate that

$$\text{dist}\,(P_\infty, P_n)^2$$
$$= \left(\frac{M_n}{D_n} - \frac{1}{\delta_r}\right)^2 + \left(\frac{1}{2}\,\frac{M_{n+2}}{D_n} - \frac{r^2}{2\delta_r}\right)^2$$
$$= \frac{1}{(2\delta_r D_n)^2}\left[4(\delta_r M_n - D_n)^2 + (\delta_r M_{n+2} - r^2 D_n)^2\right]$$
$$= \frac{1}{(2\delta_r D_n)^2}\left[(\text{Tr}\,(2s^n\beta(\delta_r - \delta_s)))^2 + (\text{Tr}\,(s^n\beta(\delta_r s^2 - \delta_s r^2)))^2\right]$$
$$\leq \frac{1}{(2.232\delta_r r^n)^2}|s^n\beta|^2 E$$
$$= \left(\frac{|\beta|\sqrt{E}}{2.232\delta_r}\omega^n\right)^2 < (0.663\omega^n)^2,$$

where $E = 8|\delta_r - \delta_s|^2 + 2|\delta_r s^2 - \delta_s r^2|^2 + 2|4(\delta_r - \delta_s)^2 + (\delta_r s^2 - \delta_s r^2)^2|$. This proves the inequality (6).

We begin the proof of formula (7) by finding the distance from $P_\infty$ to $L_n$ for any $n \geq 2$. It is convenient to set

$$\begin{aligned} \rho_n &= D_{n-1}D_{n-2}\text{dist}\,(P_{n-1}, P_{n-2}) \\ &= \sqrt{(M_{n-1}D_{n-2} - D_{n-1}M_{n-2})^2 + (N_{n-1}D_{n-2} - D_{n-1}N_{n-2})^2}. \end{aligned}$$

Suppose that $n \geq 2$ and $k \geq 1$. Now by Lemma 4.3 $L_n$ is the line through $P_{n-1}$ and $P_{n-2}$, so

$$\begin{aligned} \text{dist}\,(P_{n+k}, L_n) &= \frac{1}{\text{dist}\,(P_{n-1}, P_{n-2})} \begin{vmatrix} M_{n+k}/D_{n+k} & N_{n+k}/D_{n+k} & 1 \\ M_{n-1}/D_{n-1} & N_{n-1}/D_{n-1} & 1 \\ M_{n-2}/D_{n-2} & N_{n-2}/D_{n-2} & 1 \end{vmatrix} \\ &= \frac{|\Phi_{n,k}|}{\rho_n D_{n+k}}, \end{aligned}$$

where on the first line of the above display the double vertical lines indicate the absolute value of the determinant of the matrix and where

$$\Phi_{n,k} = \begin{vmatrix} M_{n+k} & N_{n+k} & D_{n+k} \\ M_{n-1} & N_{n-1} & D_{n-1} \\ M_{n-2} & N_{n-2} & D_{n-2} \end{vmatrix}.$$

Since the sequences $(M_n)$, $(N_n)$ and $(D_n)$ all satisfy the second recursion relation (3), we have

$$(8) \qquad\qquad \Phi_{n,1} = 0 \quad \text{and} \quad \Phi_{n,2} = 4\Phi_{n,0}.$$

Also $\Phi_{n+1,0} = 4\Phi_{n,0}$ and hence, by induction,

$$(9) \qquad\qquad \Phi_{n,0} = 4^{n-2}\Phi_{2,0} = 4^{n-2}\begin{vmatrix} 0 & 2 & 6 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{4^n}{8}.$$

We next claim that for all $n \geq 2$ and $k \geq 0$,

$$(10) \qquad\qquad \Phi_{n,k} = -\frac{4^n M_{k+1}}{8}.$$

This is obvious from formula (9) if $k = 0$ since $M_1 = 1$. It is true for $k = 1$ and $k = 2$ by the formulas in (8) since $M_2 = 0$ and $M_3 = 4$. That (10) holds for all $k \geq 0$ now follows from the fact that the sequences $(M_k)_{k \geq 0}$ and $(\Phi_{n,k})_{k \geq 0}$ both satisfy the second recursion relation. Thus,

$$
\begin{aligned}
\operatorname{dist} (P_{n+k}, L_n) &= \frac{4^n M_{k+1}}{8 \rho_n D_{n+k}} \\
&= \frac{r^{k+1}\alpha + \operatorname{Tr}\left(s^{k+1}\beta\right)}{r^{n+k}\alpha\delta_r + \operatorname{Tr}\left(s^{n+k}\delta_s\beta\right)} \frac{4^n}{8\rho_n} \longrightarrow \frac{4^n r}{8 r^n \delta_r \rho_n}
\end{aligned}
$$

as $k \to \infty$. Hence by Lemma 4.4

$$
(11) \qquad \operatorname{dist} (P_\infty, L_n) = \frac{4^n r}{8 r^n \delta_r \rho_n}.
$$

We now give an upper bound for $\rho_n$. We will use the identity

$$
\operatorname{Tr}(a\Delta)\operatorname{Tr}(b\Delta) - \operatorname{Tr}(ab\Delta)\operatorname{Tr}(\Delta) = \Delta\overline{\Delta}(\overline{a} - a)(b - \overline{b}).
$$

For any nonnegative $i$, $j$, we have

$$
\begin{aligned}
&\left| M_{i+1} D_j - D_{j+1} M_i \right| \\
&= \left| \left(r^{i+1}\alpha + \operatorname{Tr}\left(s^{i+1}\beta\right)\right)\left(r^j \delta_r \alpha + \operatorname{Tr}\left(s^j \delta_s \beta\right)\right) \right. \\
&\quad \left. - \left(r^{j+1}\delta_r \alpha + \operatorname{Tr}\left(s^{j+1}\delta_s\beta\right)\right)\left(r^i\alpha + \operatorname{Tr}\left(s^i\beta\right)\right) \right| \\
&= \left| \operatorname{Tr}\left(r^{i+1}s^j \alpha\beta\delta_s + r^j s^{i+1}\alpha\beta\delta_r - r^{j+1}s^i\alpha\beta\delta_r - r^i s^{j+1}\alpha\beta\delta_s\right) \right. \\
&\quad \left. + \operatorname{Tr}\left(s^{i+1}\beta\right)\operatorname{Tr}\left(s^j\delta_s\beta\right) - \operatorname{Tr}\left(s^{j+1}\delta_s\beta\right)\operatorname{Tr}\left(s^i\beta\right) \right| \\
&= \left| \operatorname{Tr}\left(\alpha\beta(rs)^j \left(r^{i+1-j}\delta_s + s^{i-j+1}\delta_r - rs^{i-j}\delta_r - r^{i-j}s\delta_s\right)\right) \right. \\
&\quad \left. + \beta s^i \overline{\beta}\, \overline{s}^i \left(\overline{s} - s\right)\left(\delta_s\, s^{j-i} - \overline{\delta}_s \overline{s}^{j-i}\right) \right| \\
&= |rs|^j \left| \operatorname{Tr}\left(\alpha\beta\left(\frac{s}{|\overline{s}|}\right)^j \left(r^{i+1-j}\delta_s + s^{i-j+1}\delta_r - rs^{i-j}\delta_r - r^{i-j}s\delta_s\right)\right) \right|
\end{aligned}
$$

$$+ \beta\bar{\beta}\left(\bar{s} - s\right)\left(\frac{\bar{s}}{|s|}\right)^j \left(\frac{s}{r}\right)^j \left(\delta_s \bar{s}^{i-j} - \bar{\delta}_s s^{i-j}\right)\Bigg|$$

$$\leq |rs|^j \left( 2|\alpha\beta|\left|r^{i-j+1}\delta_s + s^{i-j+1}\delta_r - rs^{i-j}\delta_r - r^{i-j}s\delta_s\right|\right.$$

$$\left. + |\beta|^2 |\bar{s} - s|\left|\frac{s}{r}\right|^j \left|\delta_s \bar{s}^{i-j} - \bar{\delta}_s s^{i-j}\right|\right).$$

We apply the last inequality with $i = j = n-2$ and with $i-2 = j = n-2$ to see that for any $k \leq n-2$

$$\rho_n^2 = (M_{n-1}D_{n-2} - D_{n-1}M_{n-2})^2 + \frac{1}{4}(M_{n+1}D_{n-2} - D_{n-1}M_n)^2$$
$$\leq |rs|^{2(n-2)}\mathcal{E}_k$$

where

$$\mathcal{E}_k = \left(2|\alpha\beta|\ |r\delta_s + s\delta_r - r\delta_r - s\delta_s| + |\beta^2||\bar{s} - s|\ |\delta_s - \bar{\delta}_s|\omega^k\right)^2$$
$$+ \frac{1}{4}\left(2|\alpha\beta|\ |(4r + 4)\delta_s + (4s + 4)\delta_r - rs^2\delta_r - r^2 s\delta_s|\right.$$
$$\left. + |\beta|^2|\bar{s} - s|\ |\delta_s \bar{s}^2 - \bar{\delta}_s s^2|\ \omega^k\right)^2.$$

Thus, for any $k \leq n-2$

$$\operatorname{dist}(P_\infty, L_n)^2 \geq \left(\frac{|rs|^n r}{8\delta_r}\right)^2 \frac{\omega^{2n}}{|rs|^{2(n-2)}\mathcal{E}_k},$$

so

$$\operatorname{dist}(P_\infty, L_n) \geq \frac{2(r + 1)\omega^n}{r\delta_r \sqrt{\mathcal{E}_k}} = \frac{2}{r + 2}\ \frac{\omega^n}{\sqrt{\mathcal{E}_k}}.$$

Hence, if $n \geq 7$ we have (taking $k = 5$)

$$\operatorname{dist}(P_\infty, L_n) \geq \frac{2}{r + 2}\ \frac{\omega^n}{\sqrt{\mathcal{E}_5}} > 0.135\,\omega^n.$$

The validity of formula (7) now follows by directly checking all cases in which $n \leq 6$ using formula (11).

We now turn very directly to the main theorem 1.1 (and Remark 1.2). If the normalized vector $v = (0, x, y, 1)$ has distance $d$ from $v_\infty$, then the

vector $(x, y)$ also has distance $d$ from $P_\infty$. Using the bounds $B = 0.685$ and $G = 0.135$ from formulas (6) and (7) we can apply the main lemma 4.5 to conclude that $v$ has length $L$ satisfying

$$-2.30 < L - \mu \ln \frac{1}{d} < -0.62 + 9,$$

so that

(12) $$\qquad -5.30 < L - \left(3 + \mu \, \ln \frac{1}{d}\right) < 5.38.$$

The main theorem and Remark 1.2 follow immediately.

5.1 *Last example.* Using the second recursion relation (3) it is easy to verify that the vectors $V_n = (0, M_n, N_n, D_n)$ have length $L = n + 4$ for all $n \geq 0$. Thus, we have a family of vectors whose lengths are unbounded and easy to compute (the Tribonacci vectors give another such family). For these vectors the error in estimating length by the simple estimate is less than 1.3. Indeed, suppose $n > 1$. The simple estimate of length for $V_n$, applied to the normalization of $V_n$, is $L* = 3 + \mu \ln(1/d)$ where $d = \mathrm{dist}\,(P_\infty, P_n)$. By Lemma 4.3 $P_n$ is on $L_{n-1}$ so $d \geq \mathrm{dist}\,(P_\infty, L_{n-1})$. Hence by the inequalities (6) and (7),

$$0.135\omega^{n-1} < d < 0.685\omega^n.$$

It follows easily that

$$-1.29 < L - L* = n + 1 + \mu \ln d < 0.38.$$

For $n \leq 30$ we found no examples in which $|L - L*|$ was more than 0.9.

5.2 *Remark.* Formula (12) implies but is slightly stronger than the result in Remark 1.2. One could improve the inequality (12) of course by less crude estimates for the parameters $B$ and $G$ in the main lemma.

## APPENDIX

In this appendix we give two arguments which were referred to earlier in the paper, but which were not actually needed in the proof of

the main theorem. Combined with Lemma 3.2, the first shows that each cyclically monotone vector is equivalent to exactly one normalized vector.

**6.1 Proposition.** *If $v$, $w \in \mathbf{R}^4$ are normalized and equivalent, then $v = w$.*

*Proof.* For any cyclically monotone vector $u$, denote the entries of $u$ by $m_u$, $a_u$, $b_u$, $M_u$ where $m_u < a_u < b_u < M_u$. Call $u$ *smooth* if $a_u + b_u < m_u + M_u$ and *rough* if $a_u + b_u > m_u + M_u$. (A normalized vector may not be smooth, but it is never rough.)

That $v$ is equivalent to $w$ implies that $w$ can be obtained from $v$ by applying a sequence of operators of the types $R$, $S$, $M_e$ ($e > 0$), $A_f$, and $M_{-1}$. Let $I$ denote the identity map on $\mathbf{R}^4$. $R$ and $S$ generate a dihedral group $\mathcal{G}$ whose elements commute with all other generators above. Using this fact and the additional relations

$$M_{-1}A_f = A_{-f}M_{-1}, \ M_{-1}M_e = M_eM_{-1}, \ M_{-1}^2 = I$$

and

$$M_eA_f = A_{fe}M_e,$$

we can write $w = M_eA_f\Delta M_{-1}^i v$ where $e > 0$, $\Delta \in \mathcal{G}$, and $i \in \{0,1\}$. If $v$ is smooth, then we must have $i = 0$ since $M_{-1}$ interchanges smoothness and roughness, while $M_e$, $A_f$ and $\Delta$ preserve them, and since $w$ cannot be rough. On the other hand, if $v$ is not smooth, then we have $a_v + b_v = 1$ and hence $M_{-1}v = A_{-1}Rv$, so without loss of generality in this case we may also suppose $i = 0$. Also $\Delta = I$ since $M_e^{-1}$ and $A_f^{-1}$ preserve monotonicity and all elements of $\mathcal{G}$ except $I$ destroy it. Then $f = 0$ (otherwise $m_w = ef \neq 0$) and finally $e = 1$ (otherwise $M_w = e \neq 1$). Therefore, $v = w$.  □

We end this section by sketching a short proof (promised at the end of Section 1) of Lotan's theorem [**3**] that *up to equivalence there is only one vector of infinite length.* This result and the above proposition together show that $v_\infty$ is the *unique normalized vector of infinite length.* (That $v_\infty$ has infinite length follows from the easily verified identity

$$T(v_\infty) = (\lambda, \lambda, \lambda, \lambda) + (1 - \lambda)v_\infty$$

where $\lambda = (6 - r^2)/2$, since this identity implies that for all $n \geq 1$,

$$T^n(v_\infty) = (1 - \lambda)^{n-1}(\lambda, \lambda, \lambda, \lambda) + (1 - \lambda)^n v_\infty \neq 0.)$$

Now suppose that $v$ is any normalized vector of infinite length. Then for all $k \geq 0$ the vector $U^k(v)$ is monotone (Lemma 3.5), so $U^k(v) = T^k(v)$ for all $k \geq 0$. The complex eigenvalues of $U$ have the form $0$, $t$, $u$, $\bar{u}$ where $t$ is real and strictly between $0$ and $1$ and $u$ is nonreal with modulus greater than 1. Let corresponding eigenvectors be $v_1$, $v_2$ (both in $\mathbf{R}^4$), $v_3$, and $v_4 = \overline{v_3}$, respectively. Since the eigenvalues are distinct, the $v_i$ are a basis for $\mathbf{C}^4$, so we can write $v = av_1 + bv_2 + cv_3 + \bar{c}\,\overline{v_3}$ for some $a$, $b$ (both in $\mathbf{R}$), and $c \in \mathbf{C}$. For all $n \geq 1$ we have $T^n v = b\,t^n v_2 + cu^n v_3 + \bar{c}\,\bar{u}^n\overline{v_3}$. Now on the one hand $(T^n(v))_{n \geq 0}$ is bounded (no coordinate of any $T^n(v)$ is larger than the maximal coordinate of $v$), and, on the other hand, $t^n \to 0$ and $|u|^n = |\bar{u}|^n \to \infty$ as $n \to \infty$. It follows that $c = 0$ and so $v = av_1 + bv_2$. (The basic observation here is that if $e \in \mathbf{C}$, then $\mathrm{Tr}\,(eu^n)$ can be bounded only if $e = 0$.) Since clearly $b \neq 0$, $v$ is equivalent to $v_2$. Thus, all vectors of infinite length are equivalent.

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