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COUNTING GENERALIZED ORDERS ON NOT NECESSARILY FORMALLY REAL FIELDS

RON BROWN

ABSTRACT. The set of classical orderings of a field compatible with a given place from the field to the real numbers is known to be bijective with the set of homomorphisms from the value group of the place into the two element group. This fact is generalized here to the set of "generalized orders" compatible with an "extended absolute value," i.e., an absolute value allowed to take the value ∞. The set of extensions to a field F of a given generalized order on a subfield of F is computed and this computation is applied to count the number of such extensions that arise from finite degree field extensions of formally p-adic fields.

1. Introduction. The theory of formally p-adic fields had its origin in Ax and Kochen's "best possible" solution of a conjecture of Artin [**1, 13**]; its further development very much used the theory of formally real fields as a model and inspiration [**9, 12**]. That there are analogies between the two theories should not be surprising; after all, a field is formally real or formally p-adic if and only if it admits a place into the field of real numbers $\mathbf{R}$ or the field of p-adic numbers $\mathbf{Q}_p$, and $\mathbf{R}$ and the fields $\mathbf{Q}_p$ are simply the completions of the rational field $\mathbf{Q}$ at its nontrivial absolute values. These parallels suggest the possibility of a common theory which applies to fields admitting a place into a specific field of characteristic zero which, like $\mathbf{R}$ or $\mathbf{Q}_p$, comes equipped with a specific absolute value. Here is an example of a result in this direction. One of the major theorems of the Artin-Schreier theory of formally real fields is the fact that the set of orderings of a formally real field is naturally bijective with the set of real closures of the field; an equivalent version of this result says that the set of orderings P of a field compatible with a place τ into $\mathbf{R}$, i.e., with $\tau(P) \geq 0$, is naturally bijective with the set of real closures of the field admitting a place into

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$\mathbf{R}$ extending τ. In [**3**] notions of "generalized orders" compatible with a place into a field equipped with an absolute value and "closures" with respect to such places are introduced, and the above theorem of Artin-Schreier is generalized to this context. Another example. The theory of formally real fields grew out of Artin's solution of Hilbert's 17th problem. Lang's classic proof of Artin's theorem [**10**] depends on an existence theorem for rational places on algebraic function fields over real closed fields. In [**6**] there is an extension of this theorem to fields closed with respect to any place into a field equipped with an absolute value. When specialized to places into $\mathbf{Q}_p$, this extension yields results of Prestel and Roquette [**12**, Section 7.1] on p-adically closed fields.

In this paper we consider a generalization in the above spirit of a much more modest result. It is a very useful fact that the set X of orderings P on a field F which are compatible with a place τ from F to $\mathbf{R}$ is bijective with the set of homomorphisms from the value group Γ of τ to the two element group $\mathbf{Z}^\bullet = \{\pm 1\}$. In general there is no such bijection which is canonical. However, it was shown in the first volume of this journal [**2**] that there is a canonical map

$$X \times X \longrightarrow \mathrm{Hom}\,(\Gamma, \mathbf{Z}^\bullet) \tag{1}$$

which is bijective in each variable. In Section 2 we will extend this result to the generalized orders compatible with a place into a field equipped with an absolute value. We also show there how the discussion can be formulated in terms of "extended absolute values," i.e., absolute values that map into the extended real numbers $\mathbf{R} \cup \{\infty\}$. This concept efficiently captures the idea of a place into a field equipped with a specific absolute value. In Section 3 we give a relative version of the generalization of the pairing (1) which applies to "admissible" field extensions. ("Admissible" field extensions are defined in Section 3; for the moment we mention only that formally real and formally p-adic fields are essentially admissible extensions of any of their subfields.) A criterion also appears there for when a given generalized order on a field can be lifted to a generalized order of an admissible field extension. As an application we count the set of extensions of a generalized order on a formally p-adic field F of arbitrary p-rank to a generalized order on a finite degree admissible extension of F.

Our notation is standard. We denote the group of multiplicative units of a (unitary) ring R by $R^\bullet$. Thus $\mathbf{Z}^\bullet = \{\pm 1\}$. We also let $f|S$ denote the restriction of a map f to a subset S of its domain. The letters m and n always denote nonnegative integers and p always denotes a rational prime number. The greatest common divisor of m and n is denoted here by (m, n).

2. The pairing for generalized orders. For the remainder of this paper τ will denote a place on a field F whose residue class field $\overline{F}$ has characteristic zero and $\overline{\varphi}$ will denote an absolute value on $\overline{F}$ such that τ and $\overline{\varphi}$ are not both trivial. We also let v, vF and $U = \tau^{-1}(\overline{F}^{\bullet})$ denote the valuation, value group and group of units corresponding to the place τ, respectively. For the theories of formally real and formally p-adic fields, the important examples are when the completion of $\overline{F}$ at $\overline{\varphi}$ is isomorphic to either the field of real numbers or to an algebraic extension of finite degree d of a field of p-adic numbers. In the latter case, F is called *formally p-adic of p-rank d* [**12**, page 92].

2.1 *Definition.* A $(\tau, \overline{\varphi})$-*order* on F is a sequence $G = (G(n))_{n>0}$ of subgroups of $F^{\bullet}$ such that for all positive integers m and n

(A) if m divides n, then $G(m) \supseteq G(n)$;

(B) $G(n) \supseteq \tau^{-1}(1)$;

(C) $v\,(G(n)) = vF$; and

(D) $\tau\,(U \cap G(n))$ is the topological closure of $\overline{F}^{\bullet n}$ in $\overline{F}^{\bullet}$ with respect to the topology induced by $\overline{\varphi}$.

Remark 2.7 below gives a compact criterion for the sequence $G = (G(n))_{n>0}$ to be a $(\tau, \overline{\varphi})$-order. The above definition is a bit complicated, but these $(\tau, \overline{\varphi})$-orders are naturally bijective, by the map $K \rightsquigarrow (K^n \cap F^{\bullet})_{n>0}$, with the maximal algebraic extensions K of F admitting a place $K \to \overline{K} \cup \{\infty\}$ extending τ such that $\overline{F}$ is dense in $\overline{K}$ with respect to an extension of $\overline{\varphi}$ [**3**, Theorem 1.5, page 753]. The real closures of F are such maximal extensions with respect to places into $\mathbf{R}$ and the p-adic closures are such maximal extensions with respect to places into $\mathbf{Q}_p$.

Let $X_{(\tau,\overline{\varphi})}$ denote the set of all $(\tau, \overline{\varphi})$-orders. Let M denote the multiplicative group of the completion of $\overline{F}$ with respect to $\overline{\varphi}$ and let $\widehat{M} = \varprojlim_{n>0} M/M^n$ denote the $\mathbf{Z}$-adic completion of M [**8**, page 165]. (If m divides n, then $M^m \supseteq M^n$, so we have natural maps $\theta = \theta_{m,n} : M/M^n \to M/M^m$. $\widehat{M}$ is the inverse limit of the resulting inverse system of groups.) Let $\beta_n : U \to M/M^n$ and $\pi_n : \widehat{M} \to M/M^n$ be the canonical maps (so $\beta_n(a) = \tau(a)M^n$). If m divides n, then $\beta_m = \theta\beta_n$ and $\pi_m = \theta\pi_n$.

2.2 Theorem. *There is a unique map*

$$\Phi : X_{(\tau,\overline{\varphi})} \times X_{(\tau,\overline{\varphi})} \longrightarrow \mathrm{Hom}\,(vF, \widehat{M}) \tag{2}$$

such that for all $H, G \in X_{(\tau,\overline{\varphi})}$, $n > 0$, $h \in H(n)$ and $g \in G(n)$ with $v(h) = v(g)$, we have

$$\pi_n \Phi(G, H)(v(g)) = \beta_n(g/h). \tag{3}$$

Moreover, Φ is bijective in each variable, i.e., for any $J \in X_{(\tau,\overline{\varphi})}$ both $\Phi(J, -)$ and $\Phi(-, J)$ are bijections from $X_{(\tau,\overline{\varphi})}$ to $\mathrm{Hom}\,(vF, \widehat{M})$.

The above theorem implies, for example, that if vF is divisible, then there is a unique $(\tau, \overline{\varphi})$-order since $\widehat{M}$ has no nontrivial divisible subgroups. The proof of Theorem 2.2 below will give a formula for the inverse of $\Phi(J, -)$.

2.3 *Note.* Φ is antisymmetric in the sense that, with the above notation, if $\gamma \in vF$ then $\Phi(H, G)(\gamma) = \Phi(G, H)(\gamma)^{-1} = \Phi(G, H)(-\gamma)$.

2.4 *Example.* We indicate here why the above theorem is a generalization of the Proposition of [**2**] saying that the map (1) above is bijective in each coordinate. Suppose that τ is a place from F to $\mathbf{R}$ and that $\overline{\varphi}$ is the usual absolute value on $\mathbf{R}$. For each classical ordering P of F associated with τ, i.e., with $\tau(P) \geq 0$, the sequence $(G(n))_{n>0}$ with $G(n) = F^\bullet$ if n is odd and $G(n) = P \cap F^\bullet$ if n is even is a $(\tau, \overline{\varphi})$-order and all $(\tau, \overline{\varphi})$-orders are of this type. In the notation of Theorem 2.2 we have $M \cong \mathbf{R}^\bullet$ and $\widehat{M} \cong \mathbf{Z}^\bullet$ (the natural map $\mathbf{R}^\bullet \to \widehat{\mathbf{R}^\bullet}$ must kill the divisible subgroup $\mathbf{R}^{\bullet 2}$ of $\mathbf{R}^\bullet$), so the objects X, Γ and $\mathbf{Z}^\bullet$ of formula (1) may be identified with the $X_{(\tau,\overline{\varphi})}$, vF and $\widehat{M}$ of Theorem 2.2.

For the remainder of this paper we will set $\varphi = \overline{\varphi} \circ \tau$ (the composition is understood to map to ∞ all elements of F which τ maps to ∞). The hypothesis that τ and $\overline{\varphi}$ are not both trivial is equivalent to the assertion that $\varphi(F) \neq \{0, 1\}$. The composition φ is an *extended absolute value* on F, i.e., a function into $\mathbf{R} \cup \{\infty\}$ with $\varphi(0) = 0$, $\varphi(1) = 1$, and for all $a, b \in F$, $\varphi(a+b) \leq \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ whenever the product $\varphi(a)\varphi(b)$ is defined. (The expressions $0 \cdot \infty$ and $\infty \cdot 0$ are not defined.) All extended absolute values arise as above as compositions of places and absolute values: if ψ is *any* extended absolute value on F, it can be regarded as the composition of the place τ_ψ associated with the valuation ring $\psi^{-1}(\mathbf{R})$ with the absolute value $\overline{\psi}$ that ψ induces on the residue class field of that place. Since φ determines both $\tau = \tau_\varphi$ and $\overline{\varphi}$, we will henceforth write X_φ in place of the more cumbersome $X_{(\tau, \overline{\varphi})}$ and, as in [**3**, Definition 1.2, page 752], simply refer to $(\tau, \overline{\varphi})$-orders as *$\varphi$-orders*. When necessary, we also indicate the dependence of the map Φ and the set of units U on φ by writing $\Phi = \Phi_\varphi$ and $U = U_\varphi$, respectively.

Extended absolute values appear to have been first introduced by André Weil [**14**, pages 420–421]. Their main interest for the author is the role they can play, along with their associated orders and closures, as invariants in the arithmetic study of general fields [**4, 5**]. (In the setting of general fields neither valuations nor absolute values can by themselves lead to a useful notion of an Archimedean prime spot.) The study of extended absolute values includes as special cases the study of absolute values, i.e., extended absolute values not taking the value ∞, and valuations (the valuation rings of F correspond bijectively to the extended absolute values on F with image contained in $\{0, 1, \infty\}$). A lot of valuation theory extends routinely and usefully to extended absolute values; concepts which extend include, for example, equivalence, comparability, topological completion, and ultracompletion (equivalently, maximal immediate extension); all these concepts depend only on the open unit disk $\{a \in F : \varphi(a) < 1\}$ [**5**]. We will use in this paper the theory in [**3**] of Henselizations at extended absolute values. The φ-orders on F contain the cohomological information which determines whether two Henselian algebraic admissible extensions of F with respect to φ with the same residue class field and value group are actually isomorphic; a necessary and sufficient condition for isomorphism is that some φ-order of F extend to orders of both of the Henselian extensions [**3**, Theorem 6.2, page 770].

The next two lemmas will be used in the proof of Theorem 2.2.

2.5 Lemma. *Let $n > 0$. The closures of $\overline{F}^{\bullet n}$ in $\overline{F}^{\bullet}$ and in M are $M^n \cap \overline{F}^{\bullet}$ and M^n, respectively.*

The above "closures" are taken with respect to the absolute value $\overline{\varphi}$ and its extension to M. (The absolute value $\overline{\varphi}$ extends canonically to the completion of $\overline{F}$ with respect to $\overline{\varphi}$; we also denote this extension by $\overline{\varphi}$.)

Proof. Let Y denote the closure of $\overline{F}^{\bullet n}$ in M. We prove here that $Y = M^n$ in the case that $\overline{\varphi}$ is non Archimedean and leave the rest of the argument to the reader.

Let $a \in Y$. Then there exists $c \in \overline{F}^{\bullet n}$ with $\overline{\varphi}\,((c^n/a) - 1) < \overline{\varphi}(n^2)$. Thus by Hensel's lemma [**7**, page 83] $c^n/a \in M^n$, so $a \in M^n$. Hence $Y \subseteq M^n$. The reverse inclusion follows easily from the continuity of the map $x \mapsto x^n$ from M to M. $\square$

2.6 Lemma. *For any subgroup J of $F^{\bullet}$ and any $n > 0$, $J \cap U = \tau^{-1}(M^n)$ if and only if $J \supseteq \tau^{-1}(1)$ and $\tau(J \cap U)$ is the closure of $\overline{F}^{\bullet n}$ in $\overline{F}^{\bullet}$ with respect to $\overline{\varphi}$.*

Proof. $(\Rightarrow)$. Since $\tau^{-1}(1) \subseteq \tau^{-1}(M^n) = J \cap U \subseteq J$, the conclusion follows immediately from Lemma 2.5.

$(\Leftarrow)$ By hypothesis and Lemma 2.5, $\tau(J \cap U) = M^n \cap \overline{F}^{\bullet}$, so $\tau^{-1}(M^n) \supseteq J \cap U$. But if $a \in \tau^{-1}(M^n)$, then there exists $b \in J \cap U$ with $\tau(a) = \tau(b)$, so

$$a = (a/b)\, b \in \tau^{-1}(1)(J \cap U) \subseteq J \cap U.$$

This shows that $J \cap U = \tau^{-1}(M^n)$. $\square$

2.7 *Remark.* Using the previous lemma one can show that a sequence $H = (H(n))_{n>0}$ of subgroups of $F^{\bullet}$ is a φ-order if and only if for all positive m and n, (i) $v(H(n) \cap H(m)) = vF$; and (ii) $H(n) \cap U = \tau^{-1}(M^n)$. After all, the previous lemma says (ii) is necessary; (i) is also necessary since if H is a φ-order, then $v(H(n) \cap H(m)) \supseteq v(H(nm)) = vF$. Now suppose that (i) and (ii) both hold. Then (B), (C) and (D) of Definition 2.1 are clear. Let m divide n and $a \in H(n)$. There exists $b \in H(n) \cap H(m)$ with $v(b) = v(a)$, so $a = b(a/b)$ lies in the set

$$\begin{aligned} H(m)(U \cap H(n)) &= H(m)\tau^{-1}(M^n) \subseteq H(m)\tau^{-1}(M^m) \\ &= H(m)(U \cap H(m)) = H(m), \end{aligned}$$

and hence condition (A) of Definition 2.1 is also satisfied.

We now turn directly to the

Proof of Theorem 2.2. Suppose $G, H \in X_\varphi$ and $\gamma \in vF$. By condition (C) of Definition 2.1 we can pick for each $n > 0$ elements $g_n \in G(n)$ and $h_n \in H(n)$ with $v(g_n) = \gamma = v(h_n)$. Now suppose $g \in G(n)$ and $h \in H(n)$ satisfy $v(g) = \gamma = v(h)$. By Lemma 2.5 and (D) of Definition 2.1, $\tau(g/g_n) \in M^n$ and $\tau(h_n/h) \in M^n$. Hence

$$\tau(g/h) = \tau(g/g_n)\tau(g_n/h_n)\tau(h_n/h) \in \tau(g_n/h_n)M^n.$$

Thus the quantity $\beta_n(g/h)$ in formula (3) is independent of the choice of g and h. In particular if m divides n, then $\theta(\beta_n(g_n/h_n)) = \beta_m(g_n/h_n) = \beta_m(g_m/h_m)$ since by condition (A) of Definition 2.1 we have $g_n \in G(n) \subseteq G(m)$ and, similarly, $h_n \in H(m)$. Hence there is a unique element $\alpha \in \widehat{M}$ with $\pi_n(\alpha) = \beta_n(g_n/h_n)$ for all $n > 0$. Thus formula (3) unambiguously defines a map $\Phi(G, H)$ from vF to $\widehat{M}$. That this map is indeed a homomorphism follows routinely from the fact that $v : F^\bullet \to vF$ and the maps β_n and π_n are themselves homomorphisms. It remains to show that Φ is bijective in each variable and hence, by Note 2.3, that

$$\Phi(G, -) : X_\varphi \longrightarrow \mathrm{Hom}\,(vF, \widehat{M})$$

is bijective. First suppose that $H, H' \in X_\varphi$ and $\Phi(G, H) = \Phi(G, H')$. Fix $n > 0$. In order to prove injectivity it suffices by symmetry to show that $H'(n) \subseteq H(n)$. Pick $a \in H'(n)$ and write $\gamma = v(a)$. Then with the notation above we have

$$\begin{aligned} \tau(g_n/a)M^n &= \beta_n(g_n/a) = \pi_n\Phi(G, H')(\gamma) = \pi_n\Phi(G, H)(\gamma) \\ &= \tau(g_n/h_n)M^n \end{aligned}$$

so by condition (D) of Definition 2.1 and Lemma 2.5

$$\tau(a/h_n) \in M^n \cap \overline{F}^\bullet = \tau(H(n) \cap U).$$

Hence $\tau(a/kh_n) = 1$ for some $k \in H(n) \cap U$, so by condition (B) of Definition 2.1, $a/(kh_n) \in H(n)$. Thus $a \in H(n)$, as required. Therefore $\Phi(G, -)$ is injective.

It remains to prove that $\Phi(G, -)$ is surjective. Let $f \in \mathrm{Hom}\,(vF, \widehat{M})$. For each $n > 0$ set

$$H(n) = \{au : a \in G(n), u \in U \quad \text{and} \quad \beta_n(u)\pi_n f(v(a)) = 1\}.$$

Using the equations $\theta\beta_n = \beta_m$ and $\theta\pi_n = \pi_m$, one easily checks that $H = (H(n))_{n>0}$ is a sequence of subgroups of $F^\bullet$ satisfying the condition (A) of Definition 2.1. Now consider any $\gamma \in vF$. Then $\gamma = v(a)$ for some $a \in G(n)$. There exists $w \in M$ with $\pi_n f(v(a)) = wM^n$. Since $\overline{F}^\bullet$ is dense in M, there exists by Hensel's lemma [**7**, page 83] $u \in U$ with $\tau(u)$ so close to w^{-1} that $\tau(u)w$ is close enough to 1 to be in M^n. Hence

$$\beta_n(u)\pi_n f(v(a)) = \tau(u)wM^n = 1$$

so $ua \in H(n)$ and $v(ua) = \gamma$. Thus the sequence H satisfies condition (C) of Definition 2.1. We next show that H satisfies conditions (B) and (D) of Definition 2.1. By Lemma 2.6 it suffices to show that $H(n)\cap U = \tau^{-1}(M^n)$ for all $n > 0$. First suppose that $c \in \tau^{-1}(M^n)$. Applying Lemma 2.6 to G yields that $c \in G(n) \cap U$ and so $c = c \cdot 1 \in H(n) \cap U$ since

$$\beta_n(c)\pi_n(f(v(1))) = \tau(c)M^n = 1.$$

Now suppose $c \in H(n) \cap U$, so we can write $c = au$ where $a \in G(n)$, $u \in U$ and $\beta_n(u)\pi_n f(v(a)) = 1$. Then $a \in G(n) \cap U$, so $\tau(a) \in M^n$. But also $v(a) = 0$, so $\beta_n(u) = 1$. That is, $\tau(u) \in M^n$, so indeed $\tau(c) = \tau(a)\tau(u) \in M^n \cap \overline{F}^\bullet$. Thus $H(n) \cap U = \pi^{-1}(M^n)$. This completes the proof that $H = (H(n))_{n>0}$ is a φ-order. Let $\gamma \in vF$. Since H is a φ-order, there exist $u \in U$ and $a \in G(n)$ with $v(ua) = \gamma$ and $\beta_n(u)\pi_n(f(v(a))) = 1$ (so $au \in H(n)$). Then by the definition of Φ,

$$\pi_n\Phi(G, H)(\gamma) = \beta_n(a/(au)) = \pi_n f(\gamma).$$

Since this is true for all γ and all n, we have $\Phi(G,H) = f$. This shows that $\Phi(G,-)$ is surjective and completes the proof of Theorem 2.2. $\square$

3. Extensions of φ-orders. As in the previous section φ is an extended absolute value on a field F such that $\overline{F}$ has characteristic zero. We will assume that $G = (G(n))_{n>0}$ is a φ-order and that (K,ψ) is an *admissible* extension of (F,φ); that is, ψ is an extended absolute value on the field extension K of F which restricts to φ on F and $\overline{F}$ is dense in $\overline{K}$ with respect to the topology of $\overline{\psi}$ (we make the natural identification of $\overline{F}$ with a subfield of $\overline{K}$). If $H = (H(n))_{n>0}$ is a ψ-order, we say it is an *extension* of G when $G(n) \subseteq H(n)$ for all $n > 0$, cf. [**3**, Section 4]. In this section we compute the set of all extensions of G to a ψ-order of K. We begin with a criterion for this set to be nonempty.

3.1 Theorem. *There is an extension of G to a ψ-order if and only if for all $n > 0$,*

$$\tau_\psi(U_\psi \cap (K^{\bullet n}G(n))) \subseteq M^n.$$

The above theorem implies that a classical ordering P of F lifts to an ordering of K if and only if there is a place $\sigma : K \to \mathbf{R} \cup \{\infty\}$ with $\sigma(PK^{\bullet 2}) \geq 0$. For any ψ-order $H = (H(n))_{n>0}$ we set $H \cap F = (H(n) \cap F)_{n>0}$; then $H \cap F \in X_\varphi$ [**3**, Proposition 4.3, page 765]. (One can easily show that H extends G if and only if $H \cap F = G$.) We will denote by vE the value group of the restriction of the valuation ring $\psi^{-1}(\mathbf{R})$ to any subfield E of K. This is consistent with our use of the notation vF since ψ extends φ.

Proof. Suppose that H is a ψ-order extending G. Then for all $n > 0$, $H(n) \supseteq K^{\bullet n}G(n)$ [**3**, Lemma 3.1C, page 762], so

$$\tau_\psi(U_\psi \cap (K^{\bullet n}G(n))) \subseteq \tau_\psi(U_\psi \cap H(n)) \subseteq M^n.$$

Now suppose conversely that $\tau_\psi(U_\psi \cap (K^{\bullet n}G(n))) \subseteq M^n$ for all $n > 0$. Let (K',ψ') be a Henselization of (K,ψ), cf. [**3**, Definition

2.10, page 759]; we will use the results of [**3**, Section 2] in this proof repeatedly, sometimes without explicit citation. Let $n > 0$. Suppose $a \in K'$ and $b \in G(n)$ and $a^n b \in U_{\psi'}$. There exists $c \in K$ with $a/c \in U_{\psi'}$ [**3**, Theorem 2.13, page 760]. Now K'/K and K/F are admissible extensions, so $(\overline{K'}, \overline{\psi'})$ has the same completion as $(\overline{F}, \overline{\varphi})$. By our hypothesis then

$$\tau_{\psi'}(a^n b) = \tau_{\psi'}(c^n b)(\tau_{\psi'}(a/c))^n \in M^n.$$

Thus $\tau_{\psi'}(U_{\psi'} \cap (K'^{\bullet n} G(n))) \subseteq M^n$. Since K' is an admissible extension of K, any ψ'-order restricts to a ψ-order [**3**, Proposition 4.3, page 765]. Hence without loss of generality we may assume (K, ψ) is Henselian. Then (K, ψ) contains a Henselization (F', φ') of (F, φ) and G extends to the φ'-order $G' = (F'^{\bullet n} G(n))_{n>0}$ [**3**, Proposition 4.4, page 765]. But then

$$\tau_\psi(U_\psi \cap (K^{\bullet n} F'^{\bullet n} G(n))) = \tau_\psi(U_\psi \cap (K^{\bullet n} G(n))) \subseteq M^n.$$

Hence without loss of generality we may assume that (F, φ) is Henselian.

Suppose that $n > 0$, $b \in K^\bullet$ and $b^n \in F$. Let (F_G, φ_G) denote the φ-closure of F with $F_G^m \cap F^\bullet = G(m)$ for all $m > 0$ [**3**, Theorem 1.5, page 753]. There exists $a \in G(n)$ with $b^n/a \in U_\psi$, so by hypothesis $\tau_\psi(b^n/a) \in M^n$. Since (F, φ) is Henselian, $\overline{F}$ is algebraically closed in its completion [**3**, Theorem 2.13, page 760]. Thus there exists $c \in U_\psi \cap F$ with $\tau_\psi(c^n) = \tau_\psi(b^n/a)$. But F with v is Henselian, so there exists $d \in F \cap U_\psi$ with $d^n = b^n/(ac^n)$, whence $b^n = a(cd)^n \in G(n)$. This proves that

$$E^n \cap F^\bullet = K^n \cap F^\bullet \subseteq G(n) = F_G^n \cap F^\bullet$$

where E denotes the relative algebraic closure of F in K. Now E and F_G are Henselian [**3**, Proposition 2.8, page 758 and Theorem 5.2, page 768] and hence there exists an F-homomorphism from $(E, \psi|E)$ to (F_G, φ_G) [**3**, Theorem 2.14, page 760]. Treating this homomorphism as an identification we find that $G_E := (F_G^n \cap E^\bullet)_{n>0}$ is a $\psi|E$-order extending G since for all $n > 0$

$$(F_G^n \cap E^\bullet) \cap F = F_G^n \cap F^\bullet = G(n).$$

Now choose a ψ-order $H = (H(n))_{n>0}$ and let $f = \Phi_{\psi|E}(G_E, H \cap E)$. Since $\widehat{M}$ is complete in the $\mathbf{Z}$-adic topology, therefore it is pure injective

[**8**, Theorems 38.1, 39.1, and 39.5, pages 160–164]. Also because E is algebraically closed in K, then vK/vE is torsion-free [**3**, Proposition 2.8, page 758]. Thus vE is a pure subgroup of vK, so f extends to a homomorphism $f^* : vK \to \widehat{M}$. By Theorem 2.2 there exists a ψ-order J with $\Phi_\psi(J, H) = f^*$. But then

$$\Phi_{\psi|E}(G_E, H \cap E) = f = f^*|vE = \Phi_\psi(J, H)|vE = \Phi_{\psi|E}(J \cap E, H \cap E),$$

so $G_E = J \cap E$ by the injectivity of $\Phi_{\psi|E}(-, H \cap E)$. Thus J is a ψ-order extending G_E and hence extending G. That is, G has an extension to a ψ-order. $\square$

We now show that the set of extensions of G to a ψ-order, if nonempty, is bijective with the set $\operatorname{Hom}(vK/vF, \widehat{M})$.

3.2 Theorem. *Let $X_\psi(G)$ denote the set of ψ-orders extending G. If $X_\psi(G)$ is nonempty, then Φ_ψ induces a map*

$$\Phi_G : X_\psi(G) \times X_\psi(G) \longrightarrow \operatorname{Hom}(vK/vF, \widehat{M})$$

which is bijective in each coordinate.

The hypothesized map Φ_G above is that map making the diagram

$$\begin{tikzcd} X_\psi(G) \times X_\psi(G) \arrow[r, "\Phi_G"] \arrow[d, "\alpha"] & \operatorname{Hom}(vK/vF, \widehat{M}) \arrow[d, "\beta"] \\ X_\psi \times X_\psi \arrow[r, "\Phi_\psi"] & \operatorname{Hom}(vK, \widehat{M}) \end{tikzcd}$$

commute, where α is the inclusion map and β is the injection induced by the canonical surjection $vK \to vK/vF$.

Proof. Suppose that $H = (H(s))_{s>0}$ and $I = (I(s))_{s>0}$ are in $X_\psi(G)$, that $\gamma \in vF$, and that $n > 0$. Then there exists $d \in G(n) \subseteq H(n) \cap I(n)$ with $v(d) = \gamma$. Therefore

$$\pi_n \Phi_\psi(H, I)(\gamma) = \beta_n(d/d) = 1$$

so $\Phi_\psi(H,I)(vF) = 1$. Thus Φ_ψ induces a map $X_\psi(G) \times X_\psi(G) \to \operatorname{Hom}(vK/vF, \widehat{M})$ and we have a commutative diagram

$$\begin{array}{ccc} X_\psi(G) & \xrightarrow{\Phi_G(H,-)} & \operatorname{Hom}(vK/vF, \widehat{M}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \beta} \\ X_\psi & \xrightarrow[\Phi_\psi(H,-)]{} & \operatorname{Hom}(vK, \widehat{M}) \end{array}$$

where i is the inclusion map (so $\alpha = i \times i$). Set $\delta = \Phi_\psi(H,-)$.

We claim that β and δi have the same image. We have already shown that $\delta i(I)$ kills vF and hence is in the image of β. On the other hand, suppose $f \in \operatorname{Hom}(vK/vF, \widehat{M})$. By Theorem 2.2 there exists $J = (J(s))_{s>0}$ in X_ψ with $\delta(J) = \beta(f)$. Then $J \cap F = (J(s) \cap F)_{s>0} \in X_\varphi$. Hence there exist $a \in J(n) \cap F$ and $b \in H(n) \cap F = G(n)$ with $v(a) = \gamma = v(b)$. Thus

$$1 = \pi_n(\beta(f)(\gamma)) = \pi_n(\delta(J)(\gamma)) = \beta_n(b/a) = \pi_n \Phi_\varphi(G, J \cap F)(\gamma),$$

so $\Phi_\varphi(G, J \cap F) = 1$. But then $G = J \cap F$ by Theorem 2.2, so $J \in X_\psi(G)$. Hence $\beta(f) = \delta i(J)$ is in the image of δi, proving our claim.

The bijectivity of $\Phi_G(H,-)$ follows immediately from the claim made in the above paragraph and the fact that both δi and β are injective. That $\Phi_G(-,H)$ is also bijective follows from the antisymmetry of Φ_ψ, cf. Note 2.3. □

We end this section with an application of Theorem 3.2 to finite degree extensions of formally p-adic fields of arbitrary p-rank. We continue to assume that K/F is an admissible extension and that $G \in X_\varphi$.

3.3 Corollary. *Suppose that the completion of $\overline{F}$ at $\overline{\varphi}$ is an extension of $\mathbf{Q}_p$ of degree d, i.e., that F is formally p-adic of p-rank d, and that the group vK/vF is finite, say with elementary divisors $n_1 \cdots n_s$. If G extends to a ψ-order, then the number of such extensions is $\prod_{i=1}^{s} (n_i, b)$, where b is the order of the torsion subgroup of M.*

The group vK/vF is finite of course if the field extension K/F has finite degree. If $d = 1$, i.e., F is formally p-adic, then the b above is $(2,p)(p-1)$; in general, b has the form $p^a(p^f-1)$ where p^f is the order of the residue class field of $\overline{\varphi}$ and a is maximal such that M has a p^ath root of unity.

Proof. From the known structure of M [**11**, Proposition 5.7, page 140] we deduce from [**8**, Theorem 39.8, page 165] that $\widehat{M} \cong \widehat{\mathbf{Z}} \times \mathbf{Z}/b\mathbf{Z} \times (I_p)^d$ (where I_p denotes the additive group of p-adic integers). Since $\widehat{\mathbf{Z}}$ is isomorphic to the direct product of the groups I_q as q ranges over all rational primes, the torsion subgroup of $\widehat{M}$ is cyclic of order b. By hypothesis $vK/vF \cong \bigoplus_{i=1}^{s} \mathbf{Z}/n_i\mathbf{Z}$, so by Theorem 3.2 the number of extensions of G to a ψ-order is the order of the group

$$\operatorname{Hom}\big(vK/vF, \widehat{M}\big) \cong \bigoplus_{i=1}^{s} \operatorname{Hom}\big(\mathbf{Z}/n_i\mathbf{Z}, \mathbf{Z}/b\mathbf{Z}\big);$$

hence the number of extensions is indeed $\prod_{i=1}^{s}(n_i, b)$. □

REFERENCES

1. J. Ax and S. Kochen, *Diophantine problems over local fields* I, Amer. J. Math. **87** (1965), 605–630.

2. R. Brown, *Real places and ordered fields*, Rocky Mountain J. Math. **1** (1971), 633–636.

3. ———, *Orders and order closures for not necessarily formally real fields*, J. Algebra **169** (1994), 751–774.

4. ———, *An approximation theorem for extended prime spots*, Canad. J. Math. **24** (1972), 167–184.

5. ———, *Extended prime spots and quadratic forms*, Pacific J. Math. **51** (1974), 379–395.

6. R. Brown and J. Merzel, *Rational places on algebraic function fields over generalized closed fields*, J. Algebra **176** (1995), 681–701.

7. J.W.S. Cassels and A. Frölich, *Algebraic number theory*, Thompson Book Co., Washington, D.C., 1967.

8. L. Fuchs, *Infinite Abelian groups*, Vol. 1, Academic Press, New York, 1970.

9. S. Kochen, *Integer valued rational functions over the p-adic numbers. A p-adic analogue of the theory of real fields*, American Math. Society, Providence, RI, 1969.

10. S. Lang, *The theory of real places*, Ann. Math. **57** (1953), 378–391.

11. J. Neukirch, *Algebraic number theory*, Springer-Verlag, Berlin, 1999.

12. A. Prestel and P. Roquette, *Formally p-adic fields*, Springer-Verlag, Berlin, 1984.

13. G. Terjanian, *Un contre-exemple à une conjecture d'Artin*, C.R. Acad. Sci. Paris **262** (1966), 612.

14. A. Weil, *Arithmetic on algebraic varieties*, Ann. Math. **53** (1951), 412–444.

Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822
Email address: ron@math.hawaii.edu