Cross Product

1. Overview
2. Definition of *determinants* in $\mathbb{R}^2$ and $\mathbb{R}^3$
3. Algebraic definition of cross-product:

   If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$,

   $$\mathbf{a} \times \mathbf{b} := \langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1 \rangle$$

4. Easier definition with determinants

Algebraic properties of the cross product

Theorem: For all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$ and $c \in \mathbb{R}$:

(a) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ (skew symmetry)

(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive law)

(c) $c(\mathbf{a} \times \mathbf{b}) = (c\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (c\mathbf{b})$

(d) $\mathbf{a} \times (c\mathbf{a}) = \mathbf{0}$

(e) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ (scalar triple product)

(f) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ (orthogonality $\mathbf{a}$)

(g) $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ (orthogonality to $\mathbf{b}$)

Proof: From the definitions

Examples Class

Geometric properties of the cross product

Theorem For all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$:

(a) (Important!) $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$
(where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$)

(b) $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a} \parallel \mathbf{b}$

(c) $\mathbf{a} \times \mathbf{b} = (\|\mathbf{a}\|\|\mathbf{b}\| \sin\theta)\mathbf{n}$
(where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and
$\mathbf{n}$ is a unit vector $\perp \mathbf{a}, \mathbf{b}$)

(d) $\|\mathbf{a} \times \mathbf{b}\|$ = area of the parallelogram with edges $\mathbf{a}$ and $\mathbf{b}$

(e) $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ = volume of the parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$

Proof: Class

Examples Class