Introduction to vectors

Notation: $\mathbb{R}$ = the real numbers

$\mathbb{R}^2$ = the set $\{(a, b) : a, b \in \mathbb{R}\}$ of ordered pairs of real numbers
= "the (Euclidean) plane," or "Euclidean 2-space" (relative to a coordinate system)

$\mathbb{R}^3$ = the set $\{(a, b, c) : a, b, c \in \mathbb{R}\}$ of ordered triples of real numbers
= "Euclidean 3-dimensional space" (relative to a coordinate system)

$\mathbb{R}^n$ = the set $\{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R}\}$ of ordered $n-$tuples of real numbers
= "Euclidean n-dimensional space" (relative to a coordinate system)

Goal: Establish mathematical object representing *displacements* in $\mathbb{R}^2, \mathbb{R}^3$, or even $\mathbb{R}^n$

Definition; Text: A *vector* in the plane is a pair $\langle a, b\rangle \in \mathbb{R}^2$

Similarly, a vector in 3-space is a triple $\langle a, b, c\rangle \in \mathbb{R}^3$

More generally, an *n-dimensional vector* is an n-tuple $\langle u_1, u_2, \ldots, u_n\rangle \in \mathbb{R}^n$

Note: we use the $\langle a, b\rangle$ notation instead of (a, b) to emphasize the *role* of the tuple.

Components: The values $u_1, u_2, \ldots$ are called the *components* of $\langle u_1, u_2, \ldots, u_n\rangle$

Warning: A vector, as a conceptual object, should exist independently of coordinate system. (Picture)

To say that the vector *is* this n-tuple is therefore not quite correct.

A vector is an 'ideal representation' of a displacement in the plane (space, etc.), which has *magnitude* and *direction.*

It is more honest to say:
We *assume* some coordinate system is fixed, and then 'identify' the space of n-dimensional vectors with $\mathbb{R}^n$

Can describe vectors pictorially.

Operations.

Let $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, and $c \in \mathbb{R}$ (a *scalar*)

Vector addition: $\mathbf{u} + \mathbf{v} := \langle u_1 + v_1, u_2 + v_2 \rangle$

Scalar multiplication: $c\mathbf{u} := \langle cu_1, cu_2 \rangle$

Examples (class)

These are the two operations that characterize an *abstract vector space* (Math 411)

Other operations can be defined in terms of these, eg:
$-\mathbf{u} := (-1)\mathbf{u}$
$\mathbf{u} - \mathbf{v} := \mathbf{u} + (-\mathbf{v})$

More generally we can define the same operations for $\mathbb{R}^n, \quad n \geq 3$:

Let $\mathbf{u} = \langle u_1, u_2, \cdots, u_n \rangle$, $\mathbf{v} = \langle v_1, v_2, \cdots, v_n \rangle$

Vector addition: $\mathbf{u}+\mathbf{v} := \langle u_1+v_1, u_2+v_2, \cdots, u_n+v_n \rangle$

Scalar multiplication: $c\mathbf{u} := \langle cu_1, cu_2, \cdots, cu_n \rangle$

Other operations, and special vectors.

Equality: $\langle u_1, u_2, \cdots, u_n\rangle = \langle v_1, v_2, \cdots, v_n\rangle$ provided $u_i = v_i$ for all i

Zero vector: 0 $:= \langle 0, 0, \cdots, 0\rangle$. Note that there is a different zero vector for each $\mathbb{R}^n$.

Standard component vectors: (In $\mathbb{R}^2$) $\mathbf{i} := \langle 1, 0\rangle$, $\mathbf{j} := \langle 0, 1\rangle$.

(In $\mathbb{R}^3$) $\mathbf{i} := \langle 1, 0, 0\rangle$, $\mathbf{j} := \langle 0, 1, 0\rangle$, $\mathbf{k} := \langle 0, 0, 1\rangle$.

Directions: $\mathbf{u}$ and $\mathbf{v}$ *have the same direction* if $\mathbf{u} = c\mathbf{v}$ for some $c > 0$

$\mathbf{u}$ and $\mathbf{v}$ *have opposite directions* if $\mathbf{u} = c\mathbf{v}$ for some $c < 0$

$\mathbf{u}$ and $\mathbf{v}$ are *parallel* if $\mathbf{u} = c\mathbf{v}$ or $\mathbf{v} = c\mathbf{u}$ for some $c \in \mathbb{R}$

Length: $|\langle u_1, u_2, \cdots, u_n\rangle|$ =the *length* or *norm* of $\langle u_1, u_2, \cdots, u_n\rangle$ $:= \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

(Sometimes you'll see $\|\mathbf{u}\|$ instead of $|\mathbf{u}|$.)

Examples (class)

Basic algebraic properties of vector operations

Theorem: Vector addition is commutative and associative,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

Multiplication by scalars is associative,

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

and satisfies the two distributive laws

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

(Here c, d are scalars, $\mathbf{u}, \mathbf{v}$ are vectors.)