Property A and graphs with large girth

Rufus Willett

July 14, 2011

Abstract

This note gives new examples of spaces without property A: the main result is that a sequence of graphs with all vertex degrees bounded between three and some upper bound, and with girth tending to infinity, does not have property A.

1 Introduction

Property A is a large-scale geometric (or 'coarse') analogue of amenability. The definition (see Definition 2.1 below) is due to Yu, who introduced property A as part of his work on the coarse Baum-Connes and Novikov conjectures [15]. It has since received a great deal of attention due to its implications for these and related conjectures, and its connections to operator algebra theory and geometric group theory: see [13] for a survey.

Due to the various implications of property A, it is interesting to give examples of bounded geometry metric spaces[1] that do not have this property. The main two examples are expanders (this is essentially an observation of Gromov) and box spaces associated to non-amenable groups (this is due to Guentner and Roe): see for example [12, Sections 11.3 and 11.5]. All other known bounded geometry spaces without property A contain a coarsely embedded copy of one of either an expander or a box space.

Both expanders and box spaces associated to non-amenable groups are examples of *spaces of graphs* as in the following definition. The definition below also sets out our conventions on graphs.

Definition 1.1. Let X be a connected graph, which we identify with its vertex set. Let $E(X)$ be the set of unordered edges of X (a subset of the quotient space of $X \times X$ by the equivalence relation $(x, y) \sim (y, x)$). Define the *edge metric* on X to be the metric

$$d(x, y) = \min\{n \in \mathbb{N} \mid \exists\, x = x_0, ..., x_n = y \text{ with } (x_i, x_{i+1}) \in E(X)\}.$$

[1] A metric space X has *bounded geometry* if for all $R > 0$ there exists $N_R \in \mathbb{N}$ such that for all $x \in X$ the ball or radius R about x has cardinality at most N_R.

A *space of graphs* is a disjoint union $X = \sqcup X_n$ where each X_n is a finite connected graph, and so that X is equipped with any metric d that restricts to the edge metric on each X_n and so that $d(X_n, X_m) \to \infty$ as $n^2 - m^2 \to \infty$.

Another motivation for studying spaces of graphs is the existence of various permanence properties in coarse geometry (see for example [9]), which imply that to understand many aspects of general metric spaces, it is essentially enough to understand spaces of graphs sufficiently well.

The main purpose of this note is to prove the following theorem.

Theorem 1.2. *Let $X = \sqcup X_n$ be a space of graphs with bounded geometry, and all vertices of degree at least three. Assume that girth$(X_n) \to \infty$. Then X does not have property A.*

In particular, X has infinite asymptotic dimension (see e.g. [5]), and does not have finite decomposition complexity [10].

Note that we do not assume that our graphs come from groups in some way, or have uniform degree.

There are several algorithms that can be used to produce spaces of graphs as in the statement (see for example [7, 4] and the references in those papers). In principle, the condition of girth tending to infinity is also a relatively easy one to check – one might hope to use this to show that certain spaces do not have property A by coarsely embedding graphs of arbitrarily large girth into them (the *Gromov monster groups* [8, 2] are a particularly interesting example where this has been done for *expanders* with girth tending to infinity). Another interesting connection is with the box space of Arzhantseva–Guentner–Špakula that coarsely embeds in Hilbert space but does not have property A [3]: this box space is a space of graphs with girth tending to infinity, and it seems reasonable to expect that there are other examples with similar properties.

The proof of Theorem 1.2 is very similar in spirit to that of the fact that a box space with property A must come from an amenable group. The main part of the proof is carried out in section 2, which indeed shows that a similar relationship holds between property A for a space of graphs, and a uniform amenability condition for an *asymptotically faithful* sequence of covers in the sense of [14]. Section 3 then uses standard results to show that a sequence of trees with all vertices of degree at least three cannot satisfy this uniform amenability condition, which completes the proof of Theorem 1.2.

2 Property A and asymptotic amenability for covering sequences

In this section we relate property A for a space of graphs to *asymptotic amenability* for a sequence of covers. Recall first the following definition of property A, which was shown to be equivalent to the original definition by Higson–Roe [11].

Definition 2.1. Let $X = \sqcup X_n$ be a bounded geometry space of graphs. Then X has *property A* if for all $\epsilon > 0$ there exists $S > 0$ and a function $\xi : X \to l^1(X)$, denoted $x \mapsto \xi_x$, such that:

- $\|\xi_x\|_1 = 1$ for all $x \in X$;
- ξ_x is supported in the ball of radius S about x for all $x \in X$;
- $\|\xi_x - \xi_y\|_1 < \epsilon$ for all $(x, y) \in E(X) := \sqcup E(X_n)$.

Definition 2.2. Let (X_n) be a sequence of infinite connected graphs with uniformly bounded degree. The sequence is said to be *asymptotically amenable* if for all $\epsilon > 0$ there exists $S > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ there exists $\phi_n \in l^1(X)$ such that:

- $\|\phi_n\|_1 = 1$;
- ϕ_n is supported in some ball of radius S in X_n;
- $\displaystyle\sum_{(x,y)\in E(X_n)} |\phi_n(x) - \phi_n(y)| < \epsilon.$

The following definitions come from [14, Section 2].

Definition 2.3. Let $X = \sqcup X_n$ be a space of graphs. A sequence $(\widetilde{X_n}, \pi_n)$ is said to be a *covering sequence* for X if for all n, $\pi_n : \widetilde{X_n} \to X_n$ is a Galois covering map. Usually we leave the maps π_n implicit and write $(\widetilde{X_n})$ for a covering sequence.

A covering sequence $(\widetilde{X_n})$ for X is said to be *asymptotically faithful* if for all $S > 0$ there exists $N > 0$ so that for all $n \geq N$ the covering maps $\pi_n : \widetilde{X_n} \to X_n$ are isometries when restricted to balls of radius S.

Example 2.4. Let $X = \sqcup X_n$ be a space of graphs such that $\mathrm{girth}(X_n)$ tends to infinity. Then the sequence of universal covers $(\widetilde{X_n})$ (a sequence of trees) is an asymptotically faithful covering sequence for X.

The following result is the main ingredient in the proof of Theorem 1.2.

Proposition 2.5. *Let $X = \sqcup X_n$ be a space of graphs with bounded geometry. Let $\widetilde{X} = (\widetilde{X_n}, \pi_n)$ be an asymptotically faithful covering sequence consisting of infinite graphs. Then if X has property A, the sequence $(\widetilde{X_n})$ is asymptotically amenable.*

Proof. Assume X has property A, and let $\epsilon > 0$. Then for all n suitably large, there exists $S > 0$ and a function $\xi^n : X_n \to l^1(X_n)$ with properties as in the definition of property A. Assume that the vertex degrees of X are all bounded by D (the existence of such a bound is equivalent to bounded geometry). Then we have the inequality

$$\sum_{(x,y)\in E(X_n)} \|\xi^n_x - \xi^n_y\|_1 < D|X_n|\epsilon \tag{1}$$

(using the fact that the size of the set we are summing over is at most $D|X_n|$). On the other hand, note that

$$|X_n| = \sum_{x \in X_n} \|\xi_x^n\|_1 = \sum_{x \in X_n} \sum_{z \in X_n} |\xi_x^n(z)|;$$

substituting this into line (1) and expanding the left-hand-side of that equation gives

$$\sum_{z \in X_n} \sum_{(x,y) \in E(X_n)} |\xi_x^n(z) - \xi_y^n(z)| < \sum_{z \in X_n} D\epsilon \sum_{x \in X_n} |\xi_x^n(z)|.$$

Now, this inequality implies that there exists $z_n \in X_n$ so that

$$\sum_{(x,y) \in E(X_n)} |\xi_x^n(z_n) - \xi_y^n(z_n)| < D\epsilon \sum_{x \in X_n} |\xi_x(z_n)|. \tag{2}$$

Define $\psi^n : X_n \to \mathbb{C}$ by $x \mapsto \xi_x^n(z_n)$. Then we have of course that ψ^n is an element of $l^1(X_n)$ supported in $B(z_n, S)$ and the inequality in line (2) can be rewritten as

$$\sum_{(x,y) \in E(X_n)} |\psi^n(x) - \psi^n(y)| < D\epsilon \|\psi^n\|_1. \tag{3}$$

Let now N be large enough so that ψ^n exists for all $n \geq N$, and so that $\pi_n : \widetilde{X_n} \to X_n$ is an isometry on all balls of radius greater than $S + 1$. Let $\widetilde{z_n} \in \widetilde{X_n}$ be any lift of z_n and define $\phi^n : \widetilde{X_n} \to \mathbb{C}$ by

$$\phi^n : x \mapsto \begin{cases} \psi^n(\pi_n(x))/\|\psi^n\|_1 & x \in B(\widetilde{z_n}, S) \\ 0 & \text{otherwise} \end{cases}$$

As D is a universal constant and ϵ arbitrary, the estimate in line (3) above, and the other properties of ψ^n, now show that the functions $(\phi^n)_{n \geq N}$ satisfy the requirements in the definition of asymptotic amenability. □

Example 2.6. Say $X = \sqcup X_n$ is a box space, so there exist an infinite finitely generated group Γ and nested sequence of normal subgroups $\Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots$ such that $\cap \Gamma_n = \{e\}$ and so that each X_n is the Cayley graph of Γ/Γ_n, defined with respect to some fixed generating set of Γ. Then the constant sequence with all elements Γ is an asymptotically faithful covering sequence for X, and is asymptotically amenable if and only if Γ is amenable. The proposition above thus reduces to the result of Guentner–Roe that if a box space has property A, then the original group must be amenable [12, Proposition 11.39].

Remark 2.7. In the case covered by the above example, one has an equivalence: Γ is amenable if and only if some (or any) box space associated to Γ has property A. Proposition 2.5 does not have a straightforward converse, however. For example, let Y_n be the Cayley graph of $SL(3, \mathbb{Z}/n\mathbb{Z})$ (defined with respect to some fixed generating set of $SL(3, \mathbb{Z})$) and C_n be the loop of length n (the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with respect to the generating set $\{1, -1\}$). Let then $X_n = Y_n \vee C_n$ be the wedge product of Y_n and C_n. It is not hard to see that

there is an asymptotically faithful covering sequence $(\widetilde{X_n})$ for the space of graphs $X = \sqcup X_n$, which is moreover asymptotically amenable (as containing 'long line segments'). The space of graphs X does not have property A, however, as it contains the expander $Y = \sqcup Y_n$.

3 Isoperimetry for trees

The purpose of this section is to show that an infinite sequence of trees, all of whose vertices have degree at least three, is not asymptotically amenable. The necessary ingredients are standard. However, we could not find exactly what we wanted in the literature, so provide proofs.

Definition 3.1. Let X be a graph, and F a subset of X. The *boundary* of F, denoted ∂F, is the set of vertices x in F such that there exists a vertex $y \in X \backslash F$ with $(x, y) \in E(X)$.

Lemma 3.2. *Let X be a graph with all vertices of degree at most some constant D, and let $\epsilon > 0$. Say $\phi \in l^1(X)$ is a finitely supported function of norm one such that*

$$\sum_{(x,y)\in E(X)} |\phi(x) - \phi(y)| < \epsilon.$$

Then there exists a (finite) subset F of the support of ϕ such that $|\partial F| < \epsilon |F|$.

The proof is a standard argument, often used to show that 'Reiter-type' definitions of amenability imply 'Følner-type' definitions.

Proof. Let ϕ be as in the statement for some $\epsilon > 0$. Replacing ϕ by $|\phi|$ if necessary, we may assume that ϕ is positive. As the support of ϕ is finite, there exist finite nested sets $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n$ and positive numbers $a_1, ..., a_n$ such that $a_1 + \cdots + a_n = 1$ and

$$\phi(x) = \sum_{i=1}^{n} \frac{a_i}{|F_i|} \chi_{F_i},$$

where χ_{F_i} is the characteristic function of F_i. We have then

$$\begin{aligned} \epsilon > \sum_{(x,y)\in E(X)} |\phi(x) - \phi(y)| &= \sum_{i=1}^{n} \frac{a_i}{|F_i|} \sum_{(x,y)\in E(X)} |\chi_{F_i}(x) - \chi_{F_i}(y)| \\ &\geq \sum_{i=1}^{n} \frac{a_i}{|F_i|} |\partial F_i| \end{aligned}$$

where we have used that the F_i are nested in the first equality. As the a_i sum to one, it follows that for at least one i it must be the case that $|\partial F_i| < \epsilon |F_i|$. Set $F = F_i$. □

Lemma 3.3. *Let X be a tree, all of whose vertices have degrees between 3 and some D. Then for any non-empty finite subset $F \subset X$, we have the inequality*

$$|\partial F| > \frac{1}{D-1}|F|.$$

Again, similar results are very well-known, but we could not find exactly what we needed in the literature. The following elegant argument is due to Brooks [6, Lemma 1].[2]

Proof. The lemma is clearly true if $|F| = 1$. If $|F| > 1$, then the boundary ∂F contains at least two points. Hence the statement holds trivially for F with $1 < |F| \leq D - 1$.

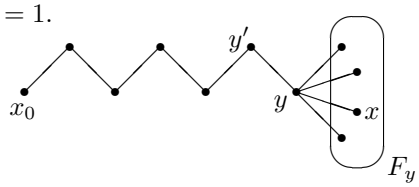
Let $F \subseteq T$ be a finite set such that $|F| > D - 1$, and fix a basepoint $x_0 \in X$. We will show that if there is a point $x \in F$ such that $d(x, x_0) \geq 2$, and if $|\partial F| \leq \frac{1}{D-1}|F|$, then we can construct a strictly smaller set F' satisfying $|\partial F'| \leq \frac{1}{D-1}|F'|$; as this inequality can easily be shown by hand to be false for subsets of the ball about x_0 of radius 1, this will lead to a contradiction. Let then $x \in F$ be such that $d(x, x_0)$ is maximal (and at least 2). Consider the (unique) vertex y in T so that $d(x, y) = 1$ and y is on the geodesic from x_0 to x.

Case 1: y is not in F. Let $F' = F \backslash \{x\}$. Then

$$\frac{|\partial F'|}{|F'|} = \frac{|\partial F| - 1}{|F| - 1} \leq \frac{|\partial F|}{|F|},$$

as long as $|\partial F| \leq |F|$.

Case 2: y is in F. Let $F_y = \{z \in X \mid d(y, z) = 1, d(z, x_0) = d(y, x_0) + 1\}$, and set $F' = F \backslash F_y$. Let y' be the unique vertex on the geodesic from x_0 to y such that $d(y, y') = 1$.



Case 2a: $y \in \partial F$ (i.e. either $y' \notin F$, or $F_y \not\subseteq F$). Then removing the points in F_y from F does not add any boundary points. Hence

$$\frac{|\partial F'|}{|F'|} = \frac{|\partial F| - |F_y \cap F|}{|F| - |F_y \cap F|} \leq \frac{|\partial F|}{|F|},$$

again assuming that $|\partial F| \leq |F|$.

Case 2b: $y \notin \partial F$ (i.e. both $y' \in F$ and $F_y \subseteq F$). Then removing the points

[2] Added in proof: Another beautiful argument can be found in [1, Proposition 5.2], as was pointed out to me by G. Arzhantseva. Strictly, both of these proofs only apply to the constant degree case - this and completeness are our reasons for providing a proof here.

in F_y from F removes $|F_y|$ points from ∂F, but also adds one (as removing F_y makes y a boundary point, which it was not before). Hence

$$\frac{|\partial F'|}{|F'|} = \frac{|\partial F| - |F_y| + 1}{|F| - |F_y|} \leq \frac{|\partial F| - 2 + 1}{|F| - (D-1)} = \frac{|\partial F| - 1}{|F| - (D-1)} \leq \frac{|\partial F|}{|F|}$$

provided $|\partial F| \leq \frac{1}{D-1}|F|$ and $|F| > D - 1$.

As these cases cover all possibilities, the argument is now complete. □

The next corollary is clear from Lemmas 3.2 and 3.3, and the definition of asymptotic amenability.

Corollary 3.4. *Let (X_n) be a sequence of trees such that the degrees of all vertices in all the X_n are between 3 and some upper bound D. Then (X_n) is not asymptotically amenable.* □

Theorem 1.2 follows from this, Example 2.4, and Proposition 2.5.

Remark 3.5. The proof of Theorem 1.2 could of course have been carried out without mentioning covering spaces at all: all our arguments could have been applied directly to the sequence (X_n) of finite graphs. We have written the piece in this way, however, in an attempt to make the connection of our work with that of Guentner–Roe on box spaces clear, and also to clarify the essential use of the 'tree-like' nature of graphs with large girth.

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