

# GHOSTBUSTING AND PROPERTY A

JOHN ROE AND RUFUS WILLET

## 1. INTRODUCTION

Let  $X$  be a metric space (we may allow  $+\infty$  as a value for some distances in  $X$ ). We say that  $X$  has *bounded geometry* if, for each  $R > 0$ , there is a natural number  $N$  such that every ball of radius  $R$  in  $X$  contains at most  $N$  points. (In particular,  $X$  is discrete.) In this paper, we will consider bounded geometry metric spaces in this sense.

Let  $X$  be such a space, and let  $\ell^2(X)$  denote the usual Hilbert space of square summable functions on  $X$  with fixed orthonormal basis  $\{\delta_x \mid x \in X\}$  of Dirac masses. Let  $\mathfrak{B}(\ell^2(X))$  denote the  $C^*$ -algebra of bounded operators on  $\ell^2(X)$ . If  $T$  is an element of  $\mathfrak{B}(\ell^2(X))$ , then  $T$  can be uniquely represented as an  $X$ -by- $X$  matrix  $(T_{xy})_{x,y \in X}$ , where

$$T_{xy} = \langle \delta_x, T\delta_y \rangle.$$

The following definitions are standard [8].

**Definition 1.1.** If  $T$  is an element of  $\mathfrak{B}(\ell^2(X))$ , then the *propagation* of  $T$  is defined to be

$$\text{Prop}(T) = \sup\{d(x, y) \mid T_{xy} \neq 0\}.$$

For each  $R \geq 0$  let  $\mathbb{C}_R[X]$  denote the collection of all operators of propagation at most  $R$ , and define

$$\mathbb{C}_u[X] := \cup_{R \in [0, \infty)} \mathbb{C}_R[X];$$

it is not difficult to see that this is a  $*$ -subalgebra of  $\mathfrak{B}(\ell^2(X))$ . We let  $C_u^*(X)$  denote its norm closure, a  $C^*$ -algebra called the *translation  $C^*$ -algebra* or *uniform Roe algebra* of  $X$ .

**Definition 1.2.** (Guoliang Yu) An operator  $T$  in  $C_u^*(X)$  is called a *ghost* if  $T_{xy} \rightarrow 0$  as  $x, y \rightarrow \infty$  in  $X$ . We denote by  $G^*(X)$  the collection of all ghost operators, which is an ideal in  $C_u^*(X)$  containing the compact operators  $\mathfrak{K}$ .

---

*Date:* April 2, 2013.

In [15], Yu introduced *property A*, an amenability property of metric spaces which has since been intensively studied (see [13] for a survey). It is easy to prove that for a property A space of bounded geometry, all ghost operators are compact. Our main objective in this paper is to show

**Theorem 1.3.** *A bounded geometry metric space without property A always admits non-compact ghosts. That is, property A is equivalent to the property “all ghosts are compact”.*

The first examples of non-compact ghosts were *projection* operators on box spaces arising from residually finite property T groups. As a corollary of our work, we see

**Theorem 1.4.** *For the bounded geometry metric space constructed by Arzhantseva, Guentner and Špakula [1], there exist non-compact ghosts, but all ghost projections are compact.*

This example embeds coarsely in Hilbert space and therefore satisfies the coarse Baum-Connes conjecture, by the work of Yu. The compactness of ghost projections is a consequence of this.

An outline of the paper is as follows. In Section 2 we review the definition of property A and of two related properties, the *operator norm localization property* (ONL) of [3] and the *uniform local amenability* (ULA) property of [2]. It is known that ONL is equivalent to property A, and ULA is implied by property A. In Section 3 we prove that the failure of ULA implies the existence of non-compact ghosts, and in Section 4 we follow a similar argument to show that the failure of ONL implies the same result. As the reader will perceive, Section 3 is therefore logically redundant, but it enables us to introduce the main idea of the proof in a more geometrically natural context. Finally, in Section 5 we investigate the existence of non-compact ghost projections.

The first author is grateful for the hospitality of the University of Hawaii during February, 2013, which made this work possible.

## 2. METRIC AMENABILITY PROPERTIES

Like other versions of amenability, property A has numerous equivalent formulations. Here is a convenient one in terms of positive definite kernels.

**Definition 2.1.** A bounded geometric metric space  $X$  has *property A* if there exists an sequence  $k_n$  of positive definite kernels on  $X \times X$  such that

- (a) Each kernel has controlled support: that is, of each  $n$  there is an  $r$  such that  $k_n(x, y) = 0$  whenever  $d(x, y) > r$ .
- (b) The sequence  $\{k_n\}$  tends to the constant 1 uniformly on each controlled set: that is, for each  $s > 0$  and  $\varepsilon > 0$  there is  $N$  such that  $1 - \varepsilon \leq k_n(x, y) \leq 1$  for all  $n > N$  and  $x, y$  such that  $d(x, y) < s$ .

It is this definition of property A that is used in the proof of the following well-known result.

**Proposition 2.2.** [8, Proposition 11.43] *On a space with property A, all ghosts are compact.*  $\square$

We will not make further direct use of the definition of property A; instead, for the remainder of the paper we will proceed via two related properties. The first of these, the *operator norm localization property*, was introduced by Chen, Tessera, Wang, and Yu [3, Section 2]. Later, Sako [12] showed that this property is equivalent to property A.

**Definition 2.3.** The space  $X$  has the *operator norm localization property* (ONL) if for all  $R \geq 0$  and  $c \in (0, 1)$  there exists  $S > 0$  such that for all  $T \in \mathbb{C}_R[X]$  of norm one there exists a norm one element  $\xi \in \ell^2(X)$  such that

$$\text{diam}(\text{Supp}(\xi)) \leq S \quad \text{and} \quad \|T\xi\| \geq c.$$

A priori, this definition of ONL is more restrictive than the original one [3, Definition 2.3], but they are equivalent by [12, Proposition 3.1].

The second property, *uniform local amenability*, was introduced by Brodzki et al. [2]. Since (as explained above) a full discussion of this property is not logically necessary to our argument, we will reformulate it in a way that is more suitable to our purposes.

**Definition 2.4.** Let  $(X_n)$  be a sequence of non-empty finite metric spaces, and let  $X = \sqcup X_n$  be the disjoint union equipped with a metric that restricts to the given metric on each  $X_n$ , and is such that  $d(X_n, X_m) > \text{diam}(X_n) + \text{diam}(X_m)$  when  $n \neq m$ . We assume that  $X$  has bounded geometry. For each  $R > 0$ , let

$$E_n^R = \{(x, y) \in X_n \times X_n \mid d(x, y) \leq R\}.$$

We say that  $X$  is a *weak expander* if there exist  $c, R > 0$  such that for all  $S > 0$  there exists  $N$  such that for all  $n \geq N$  and all  $\varphi: X_n \rightarrow \mathbb{R}$  supported in a ball of radius  $S$  we have that

$$\sum_{(x,y) \in E_n^R} |\varphi(x) - \varphi(y)| \geq c \sum_{x \in X_n} |\varphi(x)|.$$

Examples include box spaces of non-amenable groups, sequences of graphs with all vertices of degree at least 3 and girth tending to infinity [14], and expanders. In particular, the coarsely embeddable, but not property A, box space of Arzhantseva, Guentner and Špakula [1] is an example.

*Remark 2.5.* An equivalent definition of “weak expander” is given by Sako [11].

**Definition 2.6.** A bounded geometry metric space  $X$  is *uniformly locally amenable* (ULA) if it has no subspace which is a weak expander.

This is a reformulation of the definition of [2] (the equivalence of the two definitions can be proved by methods similar to those of Section 4, but we will simply use the definition above.) It is proved in [2] that property A implies uniform local amenability; in particular, no weak expander can have property A.

### 3. GHOSTS FROM WEAK EXPANDERS

In this section we will prove that a bounded geometry space that is not uniformly locally amenable has non-compact ghosts. Evidently, a ghost on a subspace of a metric space  $X$  gives rise (by “extension by zero”) to a ghost on the whole space. It therefore suffices to prove

**Proposition 3.1.** *If  $X$  is a weak expander, then  $C_u^*(X)$  contains non-compact ghost operators.*

To motivate the argument below, consider the standard example of a space with non-compact ghosts, namely a box space  $\square G$  associated to a residually finite, property T group  $G$ . The image of the Kazhdan projection under the natural homomorphism  $C^*(G) \rightarrow C_u^*(\square G)$  is a non-compact ghost. This ghost can also be regarded as the orthogonal projection on the kernel of the natural graph Laplacian  $\Delta$  on  $\square G$ , and since property T implies that the Laplacian has a spectral gap, we can also write this projection as  $f(\Delta)$  for a suitable function  $f$  supported near zero. This motivates the search for ghosts on weak expanders which also have the form  $f(\Delta)$  for  $f$  supported near zero.

Let  $X$  be a bounded geometry metric space and  $R > 0$ . Recall that the *Laplacian at scale  $R$*  is the operator  $\Delta = \Delta_R$  on  $\ell^2(X)$  defined by

$$\Delta_R: \delta_x \mapsto \sum_{y:(x,y) \in E_n^R} (\delta_x - \delta_y),$$

where the notation  $E_n^R$  has the same significance as in Definition 2.4. The operator  $\Delta_R$  has propagation  $R$ , and is bounded (since  $X$  has

bounded geometry); in particular  $\Delta_R$  is an element of  $C_u^*(X)$ . A straightforward computation shows that for any  $\xi \in \ell^2(X)$  we have

$$\langle \Delta_R \xi, \xi \rangle = \frac{1}{2} \sum_{(x,y) \in E_n^R} |\xi(x) - \xi(y)|^2,$$

and thus in particular that  $\Delta_R$  is a positive operator. We have the following lemma.

**Lemma 3.2.** *Suppose that  $X$  is a weak expander. Then there exist  $R > 0$  and  $\kappa > 0$  such that for any  $S > 0$  there exists  $N$  such that for any  $n \geq N$ , and any norm one  $\xi \in \ell^2(X_n)$  with support in a ball of radius  $S$  we have*

$$\langle \Delta_R \xi, \xi \rangle \geq \kappa.$$

*Proof.* Let  $N$  be as in Definition 2.4 for the parameter  $S$ . Let  $\xi$  be as in the statement and define  $\varphi: X_n \rightarrow \mathbb{R}$  by

$$\varphi(x) = |\xi(x)|^2.$$

The definition of weak expander implies that

$$\sum_{(x,y) \in E_n^R} |\varphi(x) - \varphi(y)| \geq c \sum_{x \in X_n} |\varphi(x)|,$$

i.e. that

$$(3.1) \quad \sum_{(x,y) \in E_n^R} ||\xi(x)|^2 - |\xi(y)|^2| \geq c \sum_{x \in X_n} |\xi(x)|^2 = c.$$

Looking at the left hand side above, we have

$$\begin{aligned} \sum_{(x,y) \in E_n^R} ||\xi(x)|^2 - |\xi(y)|^2| &= \sum_{(x,y) \in E_n^R} |(|\xi(x)| - |\xi(y)|)(|\xi(x)| + |\xi(y)|)| \\ &\leq \sqrt{\sum_{(x,y) \in E_n^R} (|\xi(x)| - |\xi(y)|)^2} \sqrt{\sum_{(x,y) \in E_n^R} (|\xi(x)| + |\xi(y)|)^2} \\ &\leq \sqrt{\sum_{(x,y) \in E_n^R} |\xi(x) - \xi(y)|^2} \sqrt{\sum_{(x,y) \in E_n^R} 2|\xi(x)|^2 + 2|\xi(y)|^2} \\ &\leq \sqrt{2\langle \Delta \xi, \xi \rangle} \sqrt{4M}, \end{aligned}$$

where  $M$  is a bound on the size of balls in  $X$  of radius  $R$ . Comparing this to line (3.1) gives the desired statement with  $\kappa = c^2/8M$ .  $\square$

*Proof of Proposition 3.1.* Let  $X$  be a weak expander, and let  $R$  and  $\kappa$  be the quantities provided by Lemma 3.2. We abbreviate  $\Delta_R$  as  $\Delta$ , and put  $m = \|\Delta\|$ . Let  $f: \mathbb{R}^+ \rightarrow [0, 1]$  be any continuous function with support in  $[0, \kappa/2]$ , and such that  $f(0) = 1$ . Since  $f(\Delta)$  majorizes

$\chi_0(\Delta)$ , which is the orthogonal projection onto the infinite-dimensional space of “ $R$ -locally constant” functions, it is clear that  $f(\Delta)$  is not compact. It suffices therefore to show that  $f(\Delta)$  is a ghost.

Let  $\varepsilon > 0$ , and let  $p$  be a polynomial such that

$$\sup_{x \in [0, m]} |f(x) - p(x)| < \varepsilon.$$

Note that the propagation of  $p(\Delta)$  is at most  $\deg(p) \cdot R$ , and that  $\|f(\Delta) - p(\Delta)\| < \varepsilon$  by the spectral theorem. Let  $S > \deg(p) \cdot R$ , and let  $N$  be as Lemma 3.2 with respect to this  $S$ . (We may assume that  $N$  is large enough that any ball of radius  $S$  whose center lies in some  $X_n$ ,  $n \geq N$ , is itself a subset of that  $X_n$ .) Let  $x$  be a point in  $X_n$  for some  $n \geq N$ . We will show that

$$\|p(\Delta)\delta_x\| < \varepsilon,$$

whence  $\|f(\Delta)\delta_x\| < 2\varepsilon$ . It follows that all the matrix entries  $\langle f(\Delta)\delta_x, \delta_y \rangle$  are bounded by  $2\varepsilon$  whenever  $x$  (or  $y$ ) lies in  $B_n$  for  $n \geq N$ . Since  $\varepsilon$  is arbitrary, this will prove that  $f(\Delta)$  is a ghost.

Consider then  $B = B(x; S) \subseteq X_n$ . Let  $P: \ell^2(X) \rightarrow \ell^2(B)$  be the orthogonal projection onto  $\ell^2(B)$ , and consider the operator  $T = P\Delta P$  as an operator on  $\ell^2(B)$ . Lemma 3.2 implies that the operator  $T$  is strictly positive, with spectrum contained in  $[\kappa, m]$ . It follows that  $f(T) = 0$ , and thus that  $\|p(T)\| < \varepsilon$ . On the other hand, for any  $k \leq \deg(p)$  the vector  $\Delta^k \delta_x$  is supported in  $B$ , whence  $\Delta^k \delta_x = T^k \delta_x$  for all such  $k$ , and thus also

$$p(\Delta)\delta_x = p(T)\delta_x.$$

Hence finally

$$\|p(\Delta)\delta_x\| = \|p(T)\delta_x\| \leq \|p(T)\| < \varepsilon,$$

completing the proof.  $\square$

*Remark 3.3.* This argument is inspired by the proof of the “partial vanishing theorem” of [10].

#### 4. GHOSTS IF ONL FAILS

In this section we will adapt the argument of Section 3 to construct non-compact ghosts for any bounded geometry metric space  $X$  that does not have the operator norm localization property. Since Sako has proved the equivalence of ONL and property A, this will complete the proof of our main result, Theorem 1.3.

We note for future reference

**Lemma 4.1.** *The operator norm localization property passes to finite unions: if  $X = Y \cup Z$ , and both  $Y$  and  $Z$  have ONL, then so does  $X$ .*

*Proof.* This is a special case of Lemma 3.3 in [4]. (Of course, granted that ONL is equivalent to property A, it also follows from the corresponding observation for property A [5].)  $\square$

The following technical lemma builds a useful sequence of operators from the failure of the operator norm localization property.

**Lemma 4.2.** *Let  $X$  be a bounded geometry metric space that does not have ONL. Then there exist  $R > 0$ ,  $\kappa < 1$ , a sequence  $(T_n)$  of operators in  $\mathbb{C}_R[X]$ , a sequence  $(B_n)$  of finite subsets of  $X$ , and a sequence  $(S_n)$  of positive real numbers such that:*

- (a)  $(S_n)$  is an increasing sequence tending to  $\infty$  as  $n$  tends to  $\infty$ ;
- (b) each  $T_n$  is positive and of norm one;
- (c) for  $n \neq m$ ,  $B_n \cap B_m = \emptyset$ ;
- (d) if  $P_n: \ell^2(X) \rightarrow \ell^2(B_n)$  denotes the orthogonal projection, then  $P_n T_n P_n = T_n$ ;
- (e) for each  $n$ , for all  $\xi \in \ell^2(X)$  satisfying

$$\|\xi\| = 1, \quad \text{diam}(\text{Supp}(\xi)) \leq S_n,$$

we have

$$\|T_n \xi_n\| \leq \kappa.$$

For simplicity, we assume in the proof that the metric on  $X$  only takes finite values: the general case can be treated similarly.

*Proof.* Fix a basepoint  $x_0$  in  $X$ , and let

$$Y = \bigsqcup_{m \text{ even}} \{x \in X \mid m^2 \leq d(x_0, x) \leq (m+1)^2\}$$

and

$$Z = \bigsqcup_{m \text{ odd}} \{x \in X \mid m^2 \leq d(x_0, x) \leq (m+1)^2\}.$$

We have then that  $X = Y \cup Z$ . By Lemma 4.1, either  $Y$  or  $Z$  does not have ONL; say without loss of generality  $Y$  does not have ONL.

Now, the negation of ONL for  $Y$  implies that there exist  $R > 0$  and  $c < 1$  such that for any  $S > 0$  there exists a norm one operator  $T \in \mathbb{C}_R[X]$  such that

$$(4.1) \quad \text{diam}(\text{Supp}(\xi)) \leq S \text{ and } \|\xi\| = 1 \text{ implies } \|T\xi\| < c.$$

We will call such an operator  $(R, c, S)$ -localized. In fact, on replacing  $T$  by  $T^*T$  (and  $R$  by  $2R$  and  $c$  by  $\sqrt{c}$ ), we see that there exist  $R > 0$  and  $c > 1$  such that for every  $S > 0$  there exists a *positive* norm

one operator which is  $(R, c, S)$ -localized. For the remainder of the proof, let  $R$  and  $c$  denote fixed quantities with this property, and let  $\kappa = 2c/(1+c) < 1$ .

Note that  $Y$  is a generalized box space — that is, a disjoint union of finite components, with the distance between components tending to infinity. It follows that there exists an  $R$ -separated decomposition

$$Y = \sqcup_{m=1}^{\infty} Y_m$$

where each  $Y_m$  is a non-empty finite subset of  $Y$  such that for  $n \neq m$ ,  $d(Y_n, Y_m) > R$ . In particular, any  $T \in \mathbb{C}_R[X]$  splits as a block diagonal sum of finite rank operators  $T = \oplus_m T^{(m)}$ ,  $T_m \in \mathfrak{B}(\ell^2(Y_m))$ , with respect to this decomposition.

We now define  $(T_n)$ ,  $(S_n)$  and  $(B_n)$  inductively as follows. Suppose that these sequences have already been defined for  $n < N$ . Choose  $M$  so large that

$$\bigcup_{n=1}^{N-1} B_n \subseteq \bigsqcup_{m \leq M} Y_m$$

and choose  $S_N$  so large that  $S_N \geq n$ ,  $S_N > S_{N-1}$  and

$$S_N > \text{diam} \left( \bigsqcup_{m \leq M} Y_m \right).$$

(In the base case  $N = 1$ , we simply set  $S_1 = 1$ .) Choose a positive norm one operator  $T \in \mathbb{C}_R[X]$  which is  $(R, c, S_N)$ -localized ((4.1)). As

$$\|T\| = \sup_{m \in \mathbb{N}} \|T^{(m)}\|_{\mathfrak{B}(\ell^2(Y_m))}$$

there exists  $m \in \mathbb{N}$  such that  $\|T^{(m)}\| > \frac{1}{2}(1+c)$ . (In particular this forces  $m > M$ .) Set

$$T_N = \frac{T^{(m)}}{\|T^{(m)}\|},$$

and note that for any  $\xi \in \ell^2(Y)$  of norm one and with  $\text{diam}(\text{Supp}(\xi)) \leq S_N$  we have that

$$\|T_N \xi\| \leq \frac{2c}{1+c} = \kappa < 1.$$

Set  $B_N = Y_m$ .

Assume then that  $(T_n)_{n \leq N-1}$ ,  $(S_n)_{n \leq N-1}$  and  $(B_n)_{n \leq N-1}$  have been defined. Let  $T$  be a positive norm one operator in  $\mathbb{C}_R[X]$  with the property in line (4.1) for  $S = S_N$ . Let  $m$  be such that  $\|T^{(m)}\| > \frac{1}{2}(1+c) > c$ , and note that by choice of  $S_N$ , this forces  $m > M$ . Set

$$T_N = \frac{T^{(m)}}{\|T^{(m)}\|},$$

and let  $B_N = Y_m$ . This completes the inductive construction.  $\square$

*Proof.* (Proof of Theorem 1.3.) Let  $X$  be a bounded geometry space without property A (or, equivalently, without ONL). Let  $R > 0$ ,  $\kappa < 1$ , and sequences  $(T_n)$ ,  $(B_n)$  and  $(S_n)$  be constructed as in Lemma 4.2 above. It follows from the construction that

$$T = \oplus T_n$$

is a positive norm one operator in  $\mathbb{C}_R[X]$ . Let now  $f: [0, 1] \rightarrow [0, 1]$  be any continuous function supported in  $[(1 + \kappa)/2, 1]$  such that  $f(1) = 1$ , and let  $f(T) \in C_u^*(X)$  denote the element given by the functional calculus. The operator  $f(T)$  is positive, norm one, and decomposes as a block diagonal sum

$$f(T) = \oplus f(T_n)$$

where each  $f(T_n)$  comes from an operator on  $\ell^2(B_n)$ .

We claim that operator  $f(T)$  so constructed is a non-compact ghost operator. To see this, note first that as each  $T_n$  is a positive, norm one, finite rank operator, 1 is an eigenvalue of  $T_n$ . It follows that  $\chi_{\{1\}}(T)$  (defined using the Borel functional calculus) is an infinite rank projection; as  $f(T) \geq \chi_{\{1\}}(T)$ , this implies that  $f(T)$  is non-compact.

It thus remains to show that  $f(T)$  is a ghost. We argue as in the proof of Proposition 3.1. It will suffice to show that for any  $\varepsilon > 0$  there exists  $N$  such that if  $n \geq N$  and  $x \in B_n$ , then  $\|f(T_n)\delta_x\| \leq 2\varepsilon$ .

Let  $p$  be a polynomial such that

$$\sup_{x \in [0, 1]} |f(x) - p(x)| < \varepsilon.$$

Note that the propagation of  $p(T)$  is at most  $\deg(p) \cdot R$ , and that

$$\|f(T) - p(T)\| = \sup_n \|f(T_n) - p(T_n)\| < \varepsilon$$

by the spectral theorem. Let  $N$  be so large that  $S_n > 2 \deg(p) \cdot R$  for all  $n \geq N$ . Let  $x$  be a point in  $B_n$  for some  $n \geq N$ . We will show that

$$\|p(T_n)\delta_x\| < \varepsilon,$$

whence  $\|f(T_n)\delta_x\| < 2\varepsilon$  as required.

Consider then  $B = B(x; \frac{1}{2}S_N)$ . Let  $P: \ell^2(X) \rightarrow \ell^2(B)$  be the orthogonal projection onto  $\ell^2(B)$ , and consider the (positive) operator  $T'_n = PT_nP$ . The fact that  $\text{diam}(B) \leq S_N \leq S_n$  implies that  $\|T'_n\| \leq c$  whence the spectrum of  $T'_n$  is contained in  $[0, c]$ . It follows that  $f(T'_n) = 0$ , and thus that  $\|p(T'_n)\| < \varepsilon$ . On the other hand, for any  $k \leq \deg(p)$  we have that  $T_n^k \delta_x$  is supported in  $B$ , whence  $T_n^k \delta_x = (T'_n)^k \delta_x$  for all such  $k$ , and thus also

$$p(T_n)\delta_x = p(T'_n)\delta_x.$$

Hence finally

$$\|p(T_n)\delta_x\| = \|p(T'_n)\delta_x\| \leq \|p(T'_n)\| < \varepsilon,$$

completing the proof of Theorem 1.3.  $\square$

*Remark 4.3.* Let  $X$  be a bounded geometry metric space without property A. Then the above construction gives rise to a ghost operator  $W = f(\Delta) \in C_u^*(X)$  that splits as a block diagonal sum  $\bigoplus_n W_n$  of norm one operators. Thus, for any two distinct subsets  $E, F$  of  $\mathbb{N}$  the operators

$$\bigoplus_{n \in E} W_n, \quad \bigoplus_{n \in F} W_n$$

are also ghosts at distance one from each other. It follows that as soon as the ghost ideal  $G^*(X)$  is not equal to the compact operators, it is not separable.

## 5. ADDITIONAL REMARKS

**5.1. Exact groups.** Let  $G$  be a discrete group (which we assume to be finitely generated in order to make contact with the metric space language of this paper). Then Ozawa [7] and Guenter-Kaminker [6] showed that the underlying coarse space of  $G$  has property A if and only if  $G$  is *exact*: that is, for any short exact sequence of  $G$ - $C^*$ -algebras

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0,$$

the corresponding sequence of (reduced) cross product algebras

$$0 \rightarrow I \rtimes G \rightarrow A \rtimes G \rightarrow B \rtimes G \rightarrow 0$$

is exact also.

Consider in particular the sequence

$$(5.1) \quad 0 \rightarrow c_0(G) \rightarrow \ell^\infty(G) \rightarrow (\ell^\infty(G)/c_0(G)) \rightarrow 0$$

of *commutative*  $G$ - $C^*$ -algebras. Taking the cross product with  $G$  we obtain the sequence

$$0 \rightarrow \mathfrak{K} \rightarrow C_u^*(|G|) \rightarrow Q \rightarrow 0,$$

where  $Q = (\ell^\infty(G)/c_0(G)) \rtimes G$ . Moreover, the first author observed in [9] that the kernel of the surjection  $C_u^*(|G|) \rightarrow Q$  is precisely the ghost ideal. Thus property A is equivalent to the statement that the cross product with  $G$  preserves exactness for *all* exact sequences of  $G$ - $C^*$ -algebras, and “all ghosts are compact” is equivalent to the statement that cross product with  $G$  preserves exactness for the single example of Equation 5.1. Our main result therefore implies

**Corollary 5.1.** *If crossed product with  $G$  preserves the exactness of the sequence given by Equation 5.1, then  $G$  is an exact group.  $\square$*

**5.2. Ghost projections.** Non-compact ghost projections are the only known source of counterexamples to the coarse Baum-Connes conjecture (in the bounded geometry setting). It is possible for  $G^*(X)$  to contain non-compact operators but not to contain any non-compact projections, however. Indeed, let  $X$  denote the box space constructed by Arzhantseva, Guentner and Špakula [1]. This space has bounded geometry and coarsely embeds into Hilbert space, but does not have property A. It follows from known results on  $K$ -theory in [14, Theorem 6.1] and [15, Theorem 1.1] that for this  $X$ , the algebra  $C_u^*(X)$  contains no non-compact ghost projections, despite containing non-compact ghosts by the results of this paper. This is Theorem 1.4.

#### REFERENCES

1. Goulmira Arzhantseva, Erik Guentner, and Ján Špakula, *Coarse non-amenability and coarse embeddings*, Geometric and Functional Analysis **22** (2012), no. 1, 22–36.
2. Jacek Brodzki, Graham A. Niblo, Jan Špakula, Rufus Willett, and Nick J. Wright, *Uniform Local Amenability*, arXiv:1203.6169 (2012).
3. Xiaoman Chen, Romain Tessera, Xianjin Wang, and Guoliang Yu, *Metric sparsification and operator norm localization*, Advances in Mathematics **218** (2008), no. 5, 1496–1511.
4. Xiaoman Chen, Qin Wang, and Xianjin Wang, *Operator norm localization property of metric spaces under finite decomposition complexity*, Journal of Functional Analysis **257** (2009), 2938–2950.
5. Marius Dadarlat and Erik Guentner, *Uniform embeddability of relatively hyperbolic groups*, Journal für die Reine und Angewandte Mathematik. [Crelle's Journal] **612** (2007), 1–15.
6. Erik Guentner and Jerome Kaminker, *Exactness and the Novikov conjecture*, Topology. An International Journal of Mathematics **41** (2002), no. 2, 411–418.
7. Narutaka Ozawa, *Amenable actions and exactness for discrete groups*, Comptes Rendus de l'Académie des Sciences. Série I. Mathématique **330** (2000), no. 8, 691–695.
8. John Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, Providence, RI, 2003.
9. ———, *Band-dominated Fredholm operators on discrete groups*, Integral Equations and Operator Theory **51** (2005), no. 3, 411–416.
10. ———, *Positive curvature, partial vanishing theorems, and coarse indices*, arXiv:1210.6100 (2012).
11. Hiroki Sako, *A generalization of expander graphs and local reflexivity of uniform Roe algebras*, arXiv:1208.5642 (2012).
12. ———, *Property A and the operator norm localization property for discrete metric spaces*, arXiv:1203.5496 (2012).

13. Rufus Willett, *Some notes on property A*, Limits of graphs in group theory and computer science, EPFL Press, Lausanne, 2009, p. 191–281.
14. Rufus Willett and Guoliang Yu, *Higher index theory for certain expanders and Gromov monster groups, I*, Advances in Mathematics **229** (2012), no. 3, 1380–1416.
15. Guoliang Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Inventiones Mathematicae **139** (2000), no. 1, 201–240.

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY  
PARK PA 16802

*E-mail address:* john.roe@psu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII AT MANOA, 2565  
MC CARTHY MALL, HONOLULU, HAWAII 96822

*E-mail address:* rufus@math.hawaii.edu