EXOTIC CROSSED PRODUCTS AND THE BAUM-CONNES CONJECTURE

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Abstract. We study general properties of exotic crossed-product functors and characterise those which extend to functors on equivariant C^*-algebra categories based on correspondences. We show that every such functor allows the construction of a descent in KK-theory and we use this to show that all crossed products by correspondence functors of K-amenable groups are KK-equivalent. We also show that for second countable groups the minimal exact Morita compatible crossed-product functor used in the new formulation of the Baum-Connes conjecture by Baum, Guentner and Willett ([4]) extends to correspondences when restricted to separable G-C^*-algebras. It therefore allows a descent in KK-theory for separable systems.

1. Introduction

The concept of a crossed product $A \rtimes_\alpha G$ by an action $\alpha : G \to \mathrm{Aut}(A)$ of a locally compact group G on a C^*-Algebra A by *-automorphisms plays a very important role in the fields of Operator Algebras and Noncommutative Geometry. Classically, there are two crossed-product constructions which have been used in the literature: the universal (or maximal) crossed product $A \rtimes_{\alpha,\mathrm{u}} G$ and the reduced (or minimal/spatial) crossed product $A \rtimes_{\alpha,\mathrm{r}} G$. The universal crossed product satisfies a universal property for covariant representations (π, u) of the underlying system (A, G, α) in such a way that every such representation integrates to a unique representation $\pi \rtimes u$ of $A \rtimes_{\alpha,\mathrm{u}} G$, while the reduced crossed product is the image of $A \rtimes_{\alpha,\mathrm{u}} G$ under the integrated form of the regular covariant representation of (A, G, α). In case $A = \mathbb{C}$ we recover the construction of the maximal and reduced group algebras $C^*(G) = \mathbb{C} \rtimes_\mathrm{u} G$ and $C^*_\mathrm{r}(G) = \mathbb{C} \rtimes_\mathrm{r} G$ of G, respectively.

The constructions of maximal and reduced crossed product extend to functors between categories with several sorts of morphisms (like equivariant *-homomorphisms or correspondences) and each one of these functors has its own special features. For instance the universal crossed-product functor always preserves short exact sequences of equivariant homomorphisms, while the reduced one only does if the group G is exact. On the other hand, the reduced crossed product always takes embeddings to embeddings, while the full one only does if the group G is amenable.

More recently (*e.g.*, see [4, 6, 8, 26, 27, 32]), there has been a growing interest in the study of exotic (or intermediate) group C^*-algebras $C^*_\mu(G)$ and crossed products $A \rtimes_{\alpha,\mu} G$, where the word "intermediate" indicates that they "lie" between the universal and reduced group algebras or crossed products. To be more precise: both

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classical crossed products can be obtained as completions of the convolution algebra $C_c(G, A)$ with respect to the universal norm $\|\cdot\|_{\mathrm{u}}$ or the reduced norm $\|\cdot\|_{\mathrm{r}} \leq \|\cdot\|_{\mathrm{u}}$, respectively. The intermediate crossed products $A \rtimes_{\alpha,\mu} G$ are completions of $C_c(G, A)$ with respect to a C^*-norm $\|\cdot\|_\mu$ such that

$$\|f\|_{\mathrm{r}} \leq \|f\|_\mu \leq \|f\|_{\mathrm{u}}$$

for all $f \in C_c(G, A)$. This is equivalent to saying that the identity on $C_c(G, A)$ induces canonical surjective *-homomorphisms

$$A \rtimes_{\alpha,\mathrm{u}} G \twoheadrightarrow A \rtimes_{\alpha,\mu} G \twoheadrightarrow A \rtimes_{\alpha,\mathrm{r}} G.$$

The main goal of this paper is to study functors $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ from the category of G-C^*-algebras into the category of C^*-algebras which assign to each G-algebra (A, α) an intermediate crossed product $A \rtimes_{\alpha,\mu} G$.

There are several reasons why researchers became interested in intermediate crossed products and group algebras. On one side intermediate group algebras provide interesting new examples of C^*-algebras attached to certain representation theoretic properties of the group. Important examples are the group algebras $C^*_{E_p}(G)$ attached to the unitary representations of G which have a dense set of matrix coefficients in $L^p(G)$, for $1 \leq p \leq \infty$, or group algebras attached to other growth conditions on their matrix coefficients. It is shown in [6, Proposition 2.11] that for every discrete group G and $1 \leq p \leq 2$ we have $C^*_{E_p}(G) = C^*_{\mathrm{r}}(G)$. But if $G = \mathbb{F}_2$ (or any discrete group which contains $\mathbb{F}_2$) then Okayasu shows in [32] that all $C^*_{E_p}(G)$ are different for $2 \leq p \leq \infty$. Very recently, a similar result has been shown by Wiersma for $\mathrm{SL}(2, \mathbb{R})$ ([37]). As one important outcome of the results of this paper we shall see that for every K-amenable group G, like $G = \mathbb{F}_2$ or $G = \mathrm{SL}(2, \mathbb{R})$, all these different group algebras are KK-equivalent. On the other hand, it has been already observed in [4, Example 6.4] that there are exotic group algebra completions $C^*_\mu(G)$ of $G = \mathbb{F}_2$ which do not have the same K-theory as $C^*_r(G)$ or $C^*(G)$. This means that the K-theory of such algebras depends not only on the structure of the group G, but also on the structure of the completion $C^*_\mu(G)$, or more precisely, on the structure of the crossed-product functor $A \mapsto A \rtimes_\mu G$. Our results will help us to understand which properties a crossed-product functor should have in order to behave well with K-theory and other constructions.

This point also brings us to another important reason for the growing interest on intermediate crossed products, which is motivated by a very recent new formulation of the Baum-Connes conjecture due to Baum, Guentner, and Willett in [4]. Recall that the Baum-Connes conjecture predicted that the K-theory of a reduced crossed product $A \rtimes_{\alpha,\mathrm{r}} G$ can be computed with the help of a canonical assembly map

$$\mathrm{as}^{\mathrm{r}}_{(A,G)} : K^{\mathrm{top}}_*(G; A) \to K_*(A \rtimes_{\alpha,\mathrm{r}} G),$$

in which $K^{\mathrm{top}}_*(G; A)$, the topological K-theory of G with coefficient A, can be computed (at least in principle) by more classical topological methods. The original Baum-Connes conjecture stated that this assembly map should always be an isomorphism. But in [23] Higson, Lafforgue and Skandalis provided counter examples for the conjecture which are based on a construction of non-exact groups due to Gromov and others. Recall that a group is called *exact* if every short exact sequence of G-algebras descents to a short exact sequence of their *reduced* crossed products.

In [4] it is shown that for every group G, there exists a crossed-product functor $(A,\alpha) \mapsto A \rtimes_{\alpha,\mathcal{E}} G$ which is minimal among all exact and (in a certain sense) Morita compatible crossed-product functors for G. Using the assembly map for the universal crossed product $A \rtimes_{\alpha,\mathrm{u}} G$ together with the canonical quotient map, we may construct an assembly map

$$\mathrm{as}^{\mu}_{(A,G)} : K^{\mathrm{top}}_*(G;A) \to K_*(A \rtimes_{\alpha,\mu} G) \tag{1.1}$$

for every intermediate crossed-product functor $\rtimes_\mu$. Using E-theory, Baum, Guentner and Willett show that all known counter-examples to the conjecture disappear if we replace reduced crossed products by $\mathcal{E}$-crossed products in the formulation of the conjecture. If G is exact, the $\mathcal{E}$-crossed product will always be the reduced one, hence the new formulation of the conjecture coincides with the classical one for such groups.

Note that the authors of [4] use E-theory instead of KK-theory since the proof requires a direct construction of the assembly map, which in the world of KK-theory would involve a descent homomorphism

$$J^{\mathcal{E}}_G : KK^G(A,B) \to KK(A \rtimes_{\mathcal{E}} G, B \rtimes_{\mathcal{E}} G)$$

and it was not clear whether such descent exists (due to exactness and Morita compatibility, it exists in E-theory). It is one consequence of our results (see §7) that a KK-descent does exist for $\rtimes_{\mathcal{E}}$ at least if we restrict to separable systems.

Indeed, in §4 we give a systematic study of crossed-product functors $\rtimes_\mu$ which are functorial for equivariant correspondences. This means that for every pair of G-algebras A, B and for every G-equivariant Hilbert $A-B$-bimodule $\mathcal{F}$ in the sense of Kasparov, there is a canonical crossed-product construction $\mathcal{F} \rtimes_\mu G$ as a Hilbert $A \rtimes_\mu G - B \rtimes_\mu G$ bimodule. We call such functors *correspondence functors*. We show in §4 that there are a number of equivalent conditions which characterise correspondence functors. For instance it turns out (see Theorem 4.9) that being a correspondence functor is equivalent to the property that functoriality extends to G-equivariant completely positive maps. But maybe the most convenient of the conditions which characterise correspondence functors is the *projection property* which requires that for every G-algebra A and every G-invariant projection $p \in \mathcal{M}(A)$, the descent $pAp \rtimes_\mu G \to A \rtimes_\mu G$ of the inclusion $pAp \hookrightarrow A$ is faithful (see Theorem 4.9). This property is easily checked in special situations and we observe that many natural examples of crossed-product functors, including all Kaliszewski-Landstad-Quigg functors (or KLQ-functors) attached to weak*-closed ideals in the Fourier-Stieltjes algebra $B(G)$ (see 2.3 for details of the construction) are correspondence functors and that correspondence functors are stable under certain manipulations which construct new functors out of old ones. In particular, for every given family of functors $\{\rtimes_{\mu_i} : i \in I\}$ there is a construction of an infimum $\rtimes_{\inf \mu}$ of this family given in [4], and it is easy to check that if all $\rtimes_{\mu_i}$ satisfy the projection property (*i.e.* are correspondence functors), then so does $\rtimes_{\inf \mu}$. It follows from this and the fact (shown in [4]) that exactness is preserved by taking the infimum of a family of functors that there exists a *minimal exact correspondence functor* $\rtimes_{\mathcal{E}_{\mathfrak{Corr}}}$ for each given locally compact group G. We show in Proposition 7.10 that this functor coincides with the minimal exact Morita compatible functor $\rtimes_{\mathcal{E}}$ of [4] if G is second countable and if we restrict the functor to separable G-algebras, as one almost always does if one discusses the Baum-Connes conjecture.

In §5 we prove that every correspondence functor will allow the construction of a descent

$$J_G^\mu : KK^G(A,B) \to KK(A \rtimes_\mu G, B \rtimes_\mu G)$$

in equivariant KK-theory, which now allows one to use the full force of equivariant KK-theory in the study of the new formulation of the Baum-Connes conjecture in [4]. Moreover, using this descent together with the deep results of [22] we can show that the assembly map (1.1) is an isomorphism whenever G is a-T-menable (or, slightly more general, if G has a γ-element equal to $1_G \in KK^G(\mathbb{C},\mathbb{C})$ in Kasparov's sense) and if $\rtimes_\mu$ is a correspondence functor on G. By a result of Tu in [36], all such groups are K-amenable, and we show in Theorem 5.6 that for every K-amenable group G and every correspondence crossed-product functor $\rtimes_\mu$ on G the quotient maps

$$A \rtimes_{\mathrm{u}} G \twoheadrightarrow A \rtimes_\mu G \twoheadrightarrow A \rtimes_{\mathrm{r}} G$$

are KK-equivalences (the proof closely follows the line of arguments given by Julg and Valette in [25, Proposition 3.4], but the main ideas are due to Cuntz in [14]). In particular, since all L^p-group algebras $C^*_{E_p}(G)$ for $p \in [1,\infty]$ are group algebras corresponding to KLQ-functors, which are correspondence functors, they all have isomorphic K-theory for a given fixed K-amenable group G. In particular this holds for interesting groups like $G = \mathbb{F}_2$ or $G = \mathrm{SL}(2,\mathbb{R})$, where $C^*_{E_p}(G)$ are all different for $p \in [2,\infty]$.

The outline of the paper is given as follows: after this introduction we start in §2 with some preliminaries on intermediate crossed-product functors and the discussion of various known examples of such functors. In §3 we isolate properties governing how crossed products behave with respect to ideals and multipliers that will be useful later. In §4 we give many useful characterisations of correspondence functors: the main result here is Theorem 4.9. Then, in §5 we show that every correspondence functor allows a KK-descent and, as an application, we use this to show the isomorphism of the assembly map for correspondence functors on a-T-menable groups and the results on K-amenability. In §6 we give a short discussion on the L^p-group algebras and applications of our results to such algebras. In §7 we show that being a correspondence functor passes to the infimum of a family of correspondence functors and we study the relation of the minimal exact correspondence functor to the functor $\rtimes_{\mathcal{E}}$ constructed in [4]. Finally in §8 we finish with some remarks and open questions.

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2. Preliminaries on crossed-product functors

Let (B,G,β) be a C^*-dynamical system. By a covariant representation (π,u) into the multiplier algebra $\mathcal{M}(D)$ of some C^*-algebra D, we understand a *-homomorphism $\pi : B \to \mathcal{M}(D)$ together with a strictly continuous *-homomorphism $u : G \to \mathcal{UM}(D)$ such that $\pi(\beta_s(b)) = u_s\pi(b)u_s^*$. If $\mathcal{E}$ is a Hilbert module (or Hilbert space) then a covariant representation on $\mathcal{E}$ will be the same as a covariant representation into $\mathcal{M}(\mathcal{K}(\mathcal{E})) \cong \mathcal{L}(\mathcal{E})$. We say that a covariant representation is *nondegenerate* if π is nondegenerate in the sense that $\pi(B)D = D$.

If (B, G, β) is a C^*-dynamical system, then $C_c(G, B)$ becomes a *-algebra with respect to the usual convolution and involution:

$$f * g(t) := \int_G f(s)\beta_s(g(s^{-1}t))\, ds \quad \text{and} \quad f^*(t) := \Delta(t)^{-1}\beta_t(f(t^{-1}))^*.$$

The full (or universal) crossed product $B \rtimes_{\beta,\mathrm{u}} G$ is the completion of $C_c(G, B)$ with respect to the universal norm $\|f\|_{\mathrm{u}} := \sup_{(\pi,u)} \|\pi \rtimes u(f)\|$ in which (π, u) runs through all covariant representations of (B, G, β) and

$$\pi \rtimes u(f) = \int_G \pi(f(s))u_s\, ds$$

denotes the integrated form of a covariant representation (π, u). By definition of this norm, each integrated form $\pi \rtimes u$ extends uniquely to a *-representation of $B \rtimes_{\beta,\mathrm{u}} G$ and this extension process gives a one-to-one correspondence between the nondegenerate covariant *-representations (π, u) of (B, G, β) and the nondegenerate *-representations of $B \rtimes_{\beta,\mathrm{u}} G$.

There is a canonical representation (ι_B, ι_G) of (B, G, β) into $\mathcal{M}(B \rtimes_{\beta,\mathrm{u}} G)$ which is determined by

$$\big(\iota_B(b) \cdot f\big)(t) = bf(t), \quad \text{and} \quad \big(\iota_G(s) \cdot f\big)(t) = \beta_s(f(s^{-1}t)),$$

for all $b \in B, s \in G$, and $f \in C_c(G, B)$. Then, for a given nondegenerate *-homomorphism $\Phi : B \rtimes_{\beta,\mathrm{u}} G \to \mathcal{M}(D)$ we have $\Phi = \pi \rtimes u$ for

$$\pi = \Phi \circ i_B \quad \text{and} \quad u = \Phi \circ i_G.$$

In this sense we get the identity $B \rtimes_{\beta,\mathrm{u}} G = i_B \rtimes i_G(B \rtimes_{\beta,\mathrm{u}} G)$.

If $B = \mathbb{C}$, then we recover the full group algebra $C^*(G) = \mathbb{C} \rtimes_{\mathrm{id},\mathrm{u}} G$ and since every nondegenerate covariant representation of $(\mathbb{C}, G, \mathrm{id})$ is of the form $(1, u)$ for some unitary representation u of G, this gives the well-known correspondence between unitary representations of G and nondegenerate representations of $C^*(G)$. In this case we denote the canonical representation of G into $\mathcal{UM}(C^*(G))$ by u_G.

For a locally compact group G, let $\lambda_G : G \to \mathcal{U}(L^2(G))$ denote the left regular representation given by $(\lambda_G(s)\xi)(t) = \xi(s^{-1}t)$ and let $M : C_0(G) \to \mathcal{L}(L^2(G))$ denote the representation of $C_0(G)$ by multiplication operators. Then, for each system (B, G, β), there is a canonical covariant homomorphism (Λ_B, Λ_G) into $\mathcal{M}(B \otimes \mathcal{K}(L^2(G)))$ given as follows: Let $\tilde{\beta} : B \to C_b(G, B) \subseteq \mathcal{M}(B \otimes C_0(G))$ be the *-homomorphism which sends $b \in B$ to $\big(s \mapsto \beta_{s^{-1}}(b)\big) \in C_b(G, B)$. Then

$$(\Lambda_B, \Lambda_G) := \big((\mathrm{id}_B \otimes M) \circ \tilde{\beta}, 1_B \otimes \lambda_G\big). \tag{2.1}$$

The *reduced crossed product* $B \rtimes_{\beta,\mathrm{r}} G$ is defined as the image of $B \rtimes_{\beta,\mathrm{u}} G$ under the integrated form $\Lambda := \Lambda_B \rtimes \Lambda_G$. Note that Λ is always faithful on the dense subalgebra $C_c(G, B)$ of $B \rtimes_{\beta,\mathrm{u}} G$, so that we may regard $B \rtimes_{\beta,\mathrm{r}} G$ also as the completion of $C_c(G, B)$ given by the norm $\|f\|_{\mathrm{r}} = \|\Lambda(f)\|$ for $f \in C_c(G, B)$. If $B = \mathbb{C}$, we get $\mathbb{C} \rtimes_{\mathrm{r}} G = \lambda_G(C^*(G)) = C^*_{\mathrm{r}}(G)$, the *reduced group C^*-algebra* of G.

2.1. **Exotic crossed products.** We shall now consider general crossed-product functors $(B, \beta) \mapsto B \rtimes_{\beta,\mu} G$ such that the μ-crossed product $B \rtimes_{\beta,\mu} G$ can be obtained as some C^*-completion of the convolution algebra $C_c(G, B)$. We always want to require that the μ-crossed products lie *between* the full and reduced crossed products in the sense that the identity map on $C_c(G, B)$ induces surjective *-homomorphisms

$$B \rtimes_{\beta,\mathrm{u}} G \twoheadrightarrow B \rtimes_{\beta,\mu} G \twoheadrightarrow B \rtimes_{\beta,\mathrm{r}} G$$

for all G-algebras (B, β). Such crossed-product functors have been studied quite recently by several authors, and we shall later recall the most prominent constructions as discussed in [4, 6, 8, 9, 26]. Note that one basic requirement will be that this construction is functorial for G-equivariant *-homomorphisms, *i.e.*, whenever we have a G-equivariant *-homomorphism $\phi : A \to B$ between two G-algebras A and B, there will be a *-homomorphism $\phi \rtimes_\mu G : A \rtimes_\mu G \to B \rtimes_\mu G$ given on the dense subalgebra $C_c(G, A)$ by $f \mapsto \phi \circ f$.

For any crossed-product functor $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ there is a canonical covariant homomorphism (i^μ_A, i^μ_G) of (A, G, α) into $\mathcal{M}(A \rtimes_{\alpha,\mu} G)$ given by the composition of the canonical covariant homomorphism (i_A, i_G) into the universal crossed product followed by the quotient map $q_{A,\mu} : A \rtimes_{\alpha,\mathrm{u}} G \to A \rtimes_{\alpha,\mu} G$. If we compose it with the quotient map $q^\mu_{\mathrm{r}} : A \rtimes_{\alpha,\mu} G \to A \rtimes_{\alpha,\mathrm{r}} G$, this will give the inclusions $(i^{\mathrm{r}}_A, i^{\mathrm{r}}_G)$ into the reduced crossed product $A \rtimes_{\alpha,\mathrm{r}} G$. Since the latter are known to be injective on A and G, the same is true i^μ_A and i^μ_G. Moreover, since $i^\mu_A : A \to \mathcal{M}(A \rtimes_{\alpha,\mu} G)$ is nondegenerate, it uniquely extends to $\mathcal{M}(A)$ (compare with [4, Lemma 4.2]).

Let us briefly discuss some particular examples of exotic crossed-product functors which have been introduced in the recent literature (*e.g.*, see the appendix of [4]). For the discussion recall that if $\pi : A \to C$ and $\rho : A \to D$ are two *-homomorphisms of a given C^*-algebra A into C^*-algebras C and D, then π is said to be *weakly contained* in ρ, denoted $\pi \preceq \rho$, if $\ker \rho \subseteq \ker \pi$, or, equivalently, if $\|\pi(a)\| \leq \|\rho(a)\|$ for all $a \in A$. The homomorphisms π and ρ are called *weakly equivalent*, denoted $\pi \approx \rho$ if their kernels coincide, *i.e.*, if $\pi \preceq \rho$ and $\rho \preceq \pi$. Similarly, if $\{\rho_i : i \in I\}$ is any collection of *-homomorphisms, we say that π is weakly contained in this collection, if $\|\pi(a)\| \leq \sup_{i \in I} \|\rho_i(a)\|$. One easily checks that in case of *-representations on Hilbert spaces, this coincides with the notion of weak containment as introduced by Fell in [19]. If G is a group and u, v are unitary representations of G, we write $u \preceq v$ if and only if this holds for their integrated forms on $C^*(G)$.

2.2. **Brown-Guentner crossed products.** The Brown-Guentner crossed products (or BG-crossed products for short) have been introduced by Brown and Guentner in [6] and can be described as follows: Let G be a locally compact group and fix any unitary representation v of G which weakly contains the regular representation λ_G. Then $C^*_v(G) := v(C^*(G))$ is a C^*-algebra which lies between the maximal and reduced group algebras of G. Clearly, any such algebra is of this form for some v and is called an *exotic group algebra* of G. Suppose now that (B, G, β) is a system. Let

$$I_{\beta,v} := \cap\{\ker(\pi \rtimes u) : u \preceq v\} \subseteq B \rtimes_{\beta,\mathrm{u}} G.$$

Then $B \rtimes_{\beta,v_{BG}} G := (B \rtimes_{\beta,\mathrm{u}} G)/I_{\beta,v}$ is called the *BG-crossed product* corresponding to v. Note that if $B = \mathbb{C}$, we recover the exotic group algebra $C^*_v(G)$ up to isomorphism.

It is very easy to check that this always defines a crossed-product functor. Note that by the definition of the notion of weak containment, the BG-crossed product only depends on the ideal $\ker v \subseteq C^*(G)$, *i.e.*, on the exotic group algebra $C^*_v(G) \cong C^*(G)/\ker v$. Notice that for the trivial coefficient $B = \mathbb{C}$, we get $\mathbb{C} \rtimes_{v_{BG}} G = C^*_v(G)$. We refer to [4, 6] for more information on BG-crossed-product functors.

Example 2.2 (*cf.*[4, Lemma A.6]). Let us consider the BG-crossed product corresponding to the regular representation $v = \lambda_G$. Although the corresponding

group algebra $C^*_v(G)$ is the reduced group algebra, the BG-functor corresponding to λ_G *does not* coincide with the reduced crossed-product functor if G is not amenable. To see this consider the action $\alpha = \mathrm{Ad}\lambda_G$ of G on $\mathcal{K}(L^2(G))$. Then the BG-crossed product $\mathcal{K}(L^2(G)) \rtimes_{\alpha,\lambda_{BG}} G$ is the full crossed product $\mathcal{K}(L^2(G)) \rtimes_{\alpha,\mathrm{u}} G \cong C^*(G) \otimes \mathcal{K}(L^2(G))$ since $(1_{C^*(G)} \otimes \mathrm{id}_{\mathcal{K}}) \rtimes (u_G \otimes \lambda)$ is an isomorphism between $\mathcal{K}(L^2(G)) \rtimes_{\alpha,\mathrm{u}} G$ and $C^*(G) \otimes \mathcal{K}(L^2(G))$ and $u_G \otimes \lambda \approx \lambda$. It therefore differs from $\mathcal{K}(L^2(G)) \rtimes_{\alpha,\mathrm{r}} G \cong C^*_{\mathrm{r}}(G) \otimes \mathcal{K}(L^2(G))$ if G is not amenable. Moreover, since α is Morita equivalent to the trivial action id on $\mathbb{C}$ via $(L^2(G), \lambda)$ and since $\mathbb{C} \rtimes_{\lambda_{BG}} G = C^*_{\mathrm{r}}(G)$, we see that BG-crossed products do not preserve Morita equivalence!

2.3. **Kaliszewski-Landstad-Quigg crossed products.** The Kaliszewski-Landstad-Quigg crossed products (or KLQ-crossed products for short) have been introduced by Kaliszewski, Landstad and Quigg in [26], and discussed since then in several papers (*e.g.*, see [4, 8, 9, 27]). The easiest way to introduce them is the following. Choose any unitary representation v of G and consider the universal covariant homomorphism (i_B, i_G) from (B, G, β) into $\mathcal{M}(B \rtimes_{\beta,\mathrm{u}} G)$. Consider the covariant representation $(i_B \otimes 1_{C^*_v(G)}, i_G \otimes v)$ of (B, G, β) into $\mathcal{M}(B \rtimes_{\beta,\mathrm{u}} G \otimes C^*_v(G))$. Let $J_{\beta,v} := \ker\big((i_B \otimes 1_{C^*_v(G)}) \rtimes (i_G \otimes v)\big) \subseteq B \rtimes_{\beta,\mathrm{u}} G$. Then the KLQ-crossed product corresponding to v is defined as

$$B \rtimes_{\beta,v_{KLQ}} G := (B \rtimes_{\beta,\mathrm{u}} G)/J_{\beta,v}.$$

A different characterisation of these KLQ-crossed products can be obtained by considering the dual coaction

$$\widehat{\beta} := (i_B \otimes 1_{C^*(G)}) \rtimes (i_G \otimes u_G) : B \rtimes_{\beta,\mathrm{u}} G \to \mathcal{M}(B \rtimes_{\beta,\mathrm{u}} G \otimes C^*(G)).$$

It follows right from the definition, that $J_{\beta,v} = \ker(\mathrm{id}_{B\rtimes G} \otimes v) \circ \widehat{\beta}$. Moreover, by the properties of minimal tensor products, this kernel only depends on the kernel $\ker v \subseteq C^*(G)$, so that the KLQ-crossed product also only depends on the quotient $C^*_v(G) \cong C^*(G)/\ker v$.

Let us see what happens if we apply this to $B = \mathbb{C}$. Using $\mathbb{C} \rtimes_{\mathrm{u}} G \cong C^*(G)$, the construction gives $\mathbb{C} \rtimes_{v_{KLQ}} G = C^*(G)/\ker(u_G \otimes v)$ which equals $C^*_v(G) = C^*(G)/\ker(v)$ if and only if v is weakly equivalent to $u_G \otimes v$. In this case we say that v *absorbs the universal representation of* G. Since $\widehat{\beta} = (i_B \otimes 1_{C^*(G)}) \rtimes (i_G \otimes u_G)$ is faithful on $B \rtimes_{\alpha,\mathrm{u}} G$, *i.e.*, it is weakly equivalent to $i_B \rtimes i_G$, it follows that $(i_B \otimes 1_{C^*_v(G)}) \rtimes (i_G \otimes v)$ is weakly equivalent to $(i_B \otimes 1_{C^*(G)} \otimes 1_{C^*_v(G)}) \rtimes (i_G \otimes u_G \otimes v)$. This follows from the properties of the minimal tensor product together with the fact that for any covariant representation (π, U) of (B, G, β), we have $(\pi \otimes 1) \rtimes (U \otimes v) = ((\pi \rtimes U) \otimes v) \circ \widehat{\beta}$. But this implies that $J_{\beta,v} = J_{\beta,u_G\otimes v}$, hence

$$B \rtimes_{\beta,v_{KLQ}} G \cong B \rtimes_{\beta,(u_G\otimes v)_{KLQ}} G.$$

Thus one can always restrict attention to representations v which absorb u_G when considering KLQ-crossed products.

It is proved in [8] that $(B, G, \beta) \mapsto B \rtimes_{\beta,v_{KLQ}} G$ is a crossed-product functor. In [9] it is proved that functoriality even extends to (equivariant) correspondences. We will have much more to say about this property in §4 below.

Another way to look at KLQ-functors is via the one-to-one correspondence between ideals in $C^*(G)$ and G-invariant (for left and right translation) weak*-closed subspaces of the Fourier-Stieltjes algebra $B(G)$, which is the algebra of all

bounded continuous functions on G which can be realised as matrix coefficients $s \mapsto \langle U_s\xi \mid \eta\rangle$ of strongly continuous unitary Hilbert-space G-representations U. Then $B(G)$ identifies with the Banach-space dual $C^*(G)^*$ if we map the function $s \mapsto \langle U_s\xi \,|\, \eta\rangle$ to the linear functional on $C^*(G)$ given by $x \mapsto \langle U(x)\xi \,|\, \eta\rangle$, where we use the same letter for U and its integrated form. It is explained in [26] that the one-to-one correspondence between closed ideals in $C^*(G)$ and G-invariant weak*-closed subspaces E in $B(G)$ is given via $E \mapsto I_E := {}^{\perp}E = \{x \in C^*(G) : f(x) = 0\ \forall f \in E\}$. Hence, we obtain all quotients of $C^*(G)$ as $C^*_E(G) := C^*(G)/I_E$ for some E. The ideal I_E will be contained in the kernel of the regular representation λ_G if and only if E contains the Fourier-algebra $A(G)$ consisting of matrix coefficients of λ_G. Moreover, the quotient map $v_E : C^*(G) \to C^*_E(G)$ absorbs the universal representation u_G if and only if E is an ideal in $B(G)$. As a consequence of this discussion we get:

Proposition 2.3. *Let G be a locally compact group. Then*

1. *there is a one-to-one correspondence between BG-crossed-product functors and G-invariant weak*-closed subspaces E of $B(G)$ which contain the Fourier algebra $A(G)$ given by sending E to the BG-functor associated to the quotient map $v_E : C^*(G) \to C^*_E(G)$.*
2. *Similarly, there is a one-to-one correspondence between KLQ-crossed-product functors and weak*-closed ideals E in $B(G)$ given by sending E to the KLQ-functor associated to the quotient map $v_E : C^*(G) \to C^*_E(G)$.*

As for BG-crossed-product functors, we get $\mathbb{C} \rtimes_{v_{KLQ}} G = C^*_v(G)$, the exotic group C^*-algebra for the trivial coefficient algebra $B = \mathbb{C}$ if v absorbs the universal representation u_G, that is, if $v = v_E$ for some G-invariant ideal in $B(G)$ as above. This is not true if E is not an ideal (compare also with [9, Proposition 2.2]).

Example 2.4. For each $p \in [2, \infty]$ we define E_p to be the weak*-closure of $B(G) \cap L^p(G)$, which is an ideal in $B(G)$. Let us write $(B, \beta) \mapsto B \rtimes_{\beta, p_{KLQ}} G$ for the corresponding KLQ-functor. It then follows from [32] that for any discrete group G which contains the free group $\mathbb{F}_2$ with two generators, the group algebras $C^*_{E_p}(G)$ are all different. Since $\mathbb{C} \rtimes_{p_{KLQ}} G = C^*_{E_p}(G)$ it follows from this that for such groups G the KLQ-crossed-product functors associated to different p are also different. Note that for $p = 2$ we just get the reduced crossed-product functor and for $p = \infty$ we get the universal crossed-product functor as the associated KLQ-functors.

The corresponding BG-crossed-product functors $(B, \beta) \mapsto B \rtimes_{\beta, p_{BG}} G$ will also be different for $2 \le p \le \infty$ because we also have $\mathbb{C} \rtimes_{p_{BG}} G = C^*_{E_p}(G)$. But Example 2.2 shows that these functors produce different results (and have different properties) for non-trivial coefficients.

2.4. **New functors from old functors.** It is shown in [4] that given any crossed-product functor $(B, \beta) \mapsto B \rtimes_{\beta,\mu} G$ and (D, δ) is any unital G-C^*-algebra, one can use D to construct a new functor $\rtimes_{\mu^\nu_D}$ by defining

$$B \rtimes_{\beta,\mu^\nu_D} G := j_B \rtimes_\mu G(B \rtimes_{\beta,\mu} G) \subseteq (B \otimes_\nu D) \rtimes_{\beta\otimes\delta,\mu} G,$$

where $\otimes_\nu$ either denotes the minimal or the maximal tensor product, and $j_B : B \to B \otimes_\nu D, j_B(b) = b \otimes 1_D$ denotes the canonical imbedding. A particularly interesting example is obtained in the case where G is discrete by taking $D = \ell^\infty(G)$ and $\rtimes_\mu = \rtimes_{\mathrm{u}}$, the universal crossed product. For general locally compact groups this

could be replaced by the algebra $UC_b(G)$ of uniformly continuous bounded functions on G: we will discuss this further in Section §8.1. For discrete G, this functor will coincide with the reduced crossed-product functor if and only if G is exact. Later, in Corollary 4.20, we shall extend the definition of $\rtimes_{\mu_D^\nu}$ to non-unital C^*-algebras D.

2.5. **A general source for (counter) examples.** The following general construction is perhaps a little unnatural. It is, however, a very useful way to produce 'counterexamples': examples of crossed-product functors that do not have good properties.

Let $\mathcal{S}$ be a collection of G-algebras, and for any given G-algebra (A, α), let $\Phi(A, \mathcal{S})$ denote the class of G-equivariant *-homomorphisms from A to an element of $\mathcal{S}$. Now define $A \rtimes_{\alpha,\mathcal{S}} G$ to be the completion of $C_c(G, A)$ for the norm

$$\|f\| := \max\Big\{\|f\|_{A\rtimes_{\alpha,\mathrm{r}}G} \quad , \quad \sup_{(\phi:A\to B)\in\Phi(A,\mathcal{S})} \|\phi\circ f\|_{B\rtimes_{\mathrm{u}}G}\Big\};$$

this is a supremum of C^*-algebra norms, so a C^*-algebra norm, and it is clearly between the reduced and maximal completions of $C_c(G, A)$.

Lemma 2.5. *The collection of completions $A\rtimes_{\mathcal{S}}G$ defines a crossed-product functor.*

Proof. This follows immediately from the fact that if $\psi : A \to B$ is any G-equivariant *-homomorphism and $\phi : B \to C$ is an element of $\Phi(B, \mathcal{S})$, then $\psi \circ \phi$ is a member of $\Phi(A, \mathcal{S})$. □

Here is an example of the sort of 'bad property' the functors above can have. We will look at some similar cases in Example 3.5 and Example 7.13 below.

Example 2.6. For a crossed-product functor $\rtimes_\mu$, write $C^*_\mu(G)$ for the exotic group algebra $\mathbb{C} \rtimes_{\mathrm{id},\mu} G$, and for a C^*-algebra A let $\mathbb{M}_2(A)$ denote the C^*-algebra of 2×2 matrices over A. If $\rtimes_\mu$ is either a KLQ- or a BG-crossed product then it is not difficult to check that

$$\mathbb{M}_2(C^*_\mu(G)) \cong \mathbb{M}_2(\mathbb{C}) \rtimes_{\mathrm{id},\mu} G. \tag{2.7}$$

However, for any non-amenable G there is a crossed product where this isomorphism fails! Indeed, in the notation above take $\mathcal{S} = \{(\mathbb{C}, \mathrm{id})\}$ and $\rtimes_\mu = \rtimes_{\mathcal{S}}$. Then $\mathbb{M}_2(C^*_\mu(G)) = \mathbb{M}_2(C^*(G))$ but $\mathbb{M}_2(\mathbb{C}) \rtimes_{\mathrm{id},\mu} G = \mathbb{M}_2(C^*_{\mathrm{r}}(G))$ (the latter follows as $\mathbb{M}_2(\mathbb{C})$ has no *-homomorphisms to $\mathbb{C}$). As G is amenable if and only if $C^*_{\mathrm{r}}(G)$ admits a non-zero finite dimensional representation, and as $C^*(G)$ always admits a non-zero one-dimensional representation, the isomorphism in line (2.7) is impossible for this crossed product and any non-amenable G.

3. The ideal property and generalised homomorphisms

In this section, we study two fundamental properties that a crossed product may or may not have: the ideal property, and functoriality under generalised homomorphisms. In the next section we will use these ideas to investigate correspondence functors.

Definition 3.1. A crossed-product functor $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ is said to be functorial for *generalised homomorphisms* if for any (possibly degenerate) G-equivariant *-homomorphism $\phi : A \to \mathcal{M}(B)$ there exists a *-homomorphism $\phi \rtimes_\mu G : A \rtimes_\mu G \to$

$\mathcal{M}(B \rtimes_\mu G)$ which is given on the level of functions $f \in C_c(G, A)$ by $f \mapsto \phi \circ f$ in the sense that $\phi \rtimes_\mu G(f)g = (\phi \circ f) * g$ for all $g \in C_c(G, B)$.

An alternative (and equivalent) definition of functoriality for generalised homomorphisms would be to ask that for every G-equivariant *-homomorphism $\phi : A \to \mathcal{M}(B)$ and every *-representation $\pi \rtimes u$ of $B \rtimes_{\beta,\mathrm{u}} G$ which factors through $B \rtimes_{\beta,\mu} G$, the representation $(\pi \circ \phi) \rtimes u$ factors through $A \rtimes_{\alpha,\mu} G$.

It follows from the definition that if a crossed-product functor is functorial for generalised homomorphisms, then it will automatically send nondegenerate G-equivariant *-homomorphisms $\phi : A \to \mathcal{M}(B)$ to nondegenerate *-homomorphisms $\phi \rtimes_\mu G : A \rtimes_\mu G \to \mathcal{M}(B \rtimes_\mu G)$, since in this case $\phi \rtimes_\mu G(C_c(G, A))C_c(G, B)$ will be inductive limit dense in $C_c(G, B)$. Note also that whenever the composition of two generalised G-equivariant morphisms $\phi : A \to \mathcal{M}(B)$ and $\psi : B \to \mathcal{M}(D)$ makes sense – *e.g.*, if the image of ϕ lies in B or if ψ is nondegenerate (in which case it uniquely extends to $\mathcal{M}(B)$), we get

$$(\psi \circ \phi) \rtimes_\mu G = (\psi \rtimes_\mu G) \circ (\phi \rtimes_\mu G),$$

since both morphisms agree on the dense subalgebra $C_c(G, A) \subseteq A \rtimes_{\alpha,\mu} G$. We shall see below that functoriality for generalised homomorphisms is equivalent to the following ideal property:

Definition 3.2. A crossed-product functor $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ satisfies the *ideal property*, if for every G-invariant closed ideal in a G-algebra A, the inclusion map $\iota : I \hookrightarrow A$ descends to an injective *-homomorphism $\iota \rtimes G : I \rtimes_\mu G \hookrightarrow A \rtimes_\mu G$.

Lemma 3.3. *Let $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ be a crossed-product functor. Then the following are equivalent:*

(1) $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ is functorial for generalised homomorphisms;
(2) $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ is functorial for nondegenerate generalised homomorphisms;
(3) $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ has the ideal property.

Proof. (1) $\Rightarrow$ (2) is trivial. (2)$\Rightarrow$ (3): Let A be a G-algebra and $I \subseteq A$ a G-invariant closed ideal. Let $\iota : I \hookrightarrow A$ denote the inclusion map. By functoriality, we get a *-homomorphism $\iota \rtimes_\mu G \colon I \rtimes_\mu G \to A \rtimes_\mu G$. Let $\phi \colon A \to \mathcal{M}(I)$ be the canonical map, which is a nondegenerate G-equivariant *-homomorphism. By (2), it induces a (nondegenerate) *-homomorphism $\phi \rtimes_\mu G \colon A \rtimes_\mu G \to \mathcal{M}(I \rtimes_\mu G)$. Notice that $\phi \circ \iota \colon I \to \mathcal{M}(I)$ is the canonical inclusion map, which is a nondegenerate generalised homomorphism, so it induces a *-homomorphism $(\phi \circ \iota) \rtimes_\mu G \colon I \rtimes_\mu G \to \mathcal{M}(I \rtimes_\mu G)$. It is then easily checked on $C_c(G, I)$ that $(\phi \circ \iota) \rtimes_\mu G$ coincides with the the canonical embedding $I \rtimes_\mu G \hookrightarrow \mathcal{M}(I \rtimes_\mu G)$. In particular, $(\phi \circ \iota) \rtimes_\mu G = (\phi \rtimes_\mu G) \circ (\iota \rtimes_\mu G)$ is injective, which implies the injectivity of $\iota \rtimes_\mu G$, as desired.

(3) $\Rightarrow$ (1): Assume now that the μ-crossed-product functor has the ideal property and let A and B be G-algebras and $\phi : A \to \mathcal{M}(B)$ a G-equivariant *-homomorphism. Let $\mathcal{M}_c(B) = \{m \in \mathcal{M}(B) : s \mapsto \beta_s(m) \text{ is norm continuous}\}$ be the subalgebra of $\mathcal{M}(B)$ of G-continuous elements. Then ϕ takes values in $\mathcal{M}_c(B)$ and by functoriality we see that we get a *-homomorphism $\phi \rtimes_\mu G : A \rtimes_\mu G \to \mathcal{M}_c(B) \rtimes_\mu G$. Now by the ideal property, we also have a faithful inclusion $\iota \rtimes_\mu G : B \rtimes_\mu G \to \mathcal{M}_c(B) \rtimes_\mu G$, hence we may regard $B \rtimes_\mu G$ as an ideal in $\mathcal{M}_c(B) \rtimes_\mu G$. But this implies that there is a canonical *-homomorphism $\Phi : \mathcal{M}_c(B) \rtimes_\mu G \to \mathcal{M}(B \rtimes_\mu G)$. It is then

straightforward to check that the composition $\Phi \circ (\phi \rtimes_\mu G) : A \rtimes_\mu G \to \mathcal{M}(B \rtimes_\mu G)$ is given on functions in $C_c(G, A)$ by sending $f \in C_c(G, A)$ to $\phi \circ f \in C_c(G, \mathcal{M}(B))$ as required. □

Remark 3.4. It is easy to see directly that all KLQ- and all BG-crossed products have the ideal property.

Recall from [4] that a crossed-product functor $A \mapsto A \rtimes_\mu G$ is *exact* if it preserves short exact sequences, that is, if every short exact sequence $0 \to I \to A \to B \to 0$ of G-C^*-algebras induces a short exact sequence $0 \to I \rtimes_\mu G \to A \rtimes_\mu G \to B \rtimes_\mu G \to 0$ of C^*-algebras. In particular, all exact functors satisfy the ideal property, but the ideal property alone is far from being enough for exactness since the reduced crossed-product functor always satisfies the ideal property. By definition, the *group G is exact* if $A \mapsto A \rtimes_{\mathrm{r}} G$ is exact.

Example 3.5. Every non-amenable locally compact group admits a crossed-product functor which does not satisfy the ideal property. To see this, we use the construction of Section 2.5. Let $\mathcal{S}$ be the collection of G-algebras consisting of only $C_0([0,1))$ equipped with the trivial action id. For each G-algebra (A, α) we define $A \rtimes_{\alpha,\mathcal{S}} G$ as the completion of $C_c(G, A)$ with respect to the norm

$$\|f\|_{\mathcal{S}} := \max\{\|f\|_{\mathrm{r}}, \sup_{\phi} \|\phi \circ f\|_{\mathrm{u}}\},$$

where the supremum is taken over all G-equivariant *-homomorphisms $\phi : A \to C_0([0,1))$. As explained in Section 2.5, this is a crossed-product functor.

To see that it does not have the ideal property, consider the short exact sequence of (trivial) G-algebras

$$0 \to C_0([0,1)) \to C([0,1]) \to \mathbb{C} \to 0.$$

Observe that $A \rtimes_\mu G = A \rtimes_{\mathrm{r}} G$ for every unital G-algebra A since there is no non-zero homomorphism $A \to C_0([0,1))$. In particular, $C([0,1]) \rtimes_\mu G = C([0,1]) \rtimes_{\mathrm{r}} G \cong C([0,1]) \otimes C^*_{\mathrm{r}}(G)$. On the other hand, we obviously have $C_0([0,1)) \rtimes_\mu G = C_0([0,1)) \rtimes_{\mathrm{u}} G \cong C([0,1)) \otimes C^*(G)$. Hence, if G is not amenable, the canonical map $\iota \rtimes_\mu G : C_0([0,1)) \rtimes_\mu G \to C([0,1]) \rtimes_\mu G$ will not be injective.

For crossed-product functors which enjoy the ideal property we have the important result:

Lemma 3.6. *Let $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ be a crossed-product functor with the ideal property and let X be a locally compact Hausdorff space. Let (j_A, j_X) denote the canonical inclusions of A and $C_0(X)$ into $\mathcal{M}(A \otimes C_0(X))$, respectively and let $i_{A\otimes C_0(X)} : A \otimes C_0(X) \to \mathcal{M}\big((A \otimes C_0(X)) \rtimes_{\alpha\otimes \mathrm{id}_X,\mu} G\big)$ be the canonical map. Then*

$$\Psi_X := (j_A \rtimes_\mu G) \times (i_{A\otimes C_0(X)} \circ j_X) : A \rtimes_{\alpha,\mu} G \otimes C_0(X) \xrightarrow{\cong} (A \otimes C_0(X)) \rtimes_{\alpha\otimes \mathrm{id}_X,\mu} G$$

is an isomorphism.

Remark 3.7. We should remark that if X is compact, the result holds for any crossed-product functor $\rtimes_\mu$. The proof of this fact is given for the case $X = [0,1]$ in [4, Lemma 4.3], but the same arguments will work for arbitrary compact Hausdorff spaces as well. We will use this fact in the proof below.

Proof. Let $X_\infty = X \cup \{\infty\}$ denote the one-point compactification of X. Then the above remark implies that the lemma is true for X_∞. Consider the diagram

$$\begin{array}{ccc} (A \otimes C_0(X)) \rtimes_\mu G & \longrightarrow & (A \otimes C(X_\infty)) \rtimes_\mu G \\ {\scriptstyle \Psi_X}\big\downarrow & & {\scriptstyle \Psi_{X_\infty}}\big\downarrow{\scriptstyle \cong} \\ A \rtimes_{\alpha,\mu} G \otimes C_0(X) & \longrightarrow & A \rtimes_{\alpha,\mu} G \otimes C(X_\infty) \end{array}$$

in which the horizontal maps are induced by the canonical inclusion $C_0(X) \hookrightarrow C(X_\infty)$. One checks on elements of the form $f \otimes g$ with $f \in C_0(X)$ and $g \in C_c(G, A)$ that the diagram commutes and that Ψ_X has dense image in $A \rtimes_{\alpha,\mu} G \otimes C_0(X)$. By the ideal property, the upper horizontal map is injective, which then implies that Ψ_X is injective as well. □

4. Correspondence functors

In this section we continue our study of properties that a crossed-product functor may or may not have. One particularly important property we would like to have is that the functor extends to a functor on the G-equivariant correspondence category, in which the objects are G-algebras and the morphisms are G-equivariant correspondences. The importance of this comes from the fact that functoriality for correspondences allows the construction of a descent in equivariant KK-theory

$$J_G^\mu : KK^G(A, B) \to KK(A \rtimes_\mu G, B \rtimes_\mu G)$$

for our exotic functor $\rtimes_\mu$; we discuss this in §5. The main result of this section is Theorem 4.9, which gives several equivalent characterisations of functors that extend to the correspondence category.

We start by recalling some background about correspondences.

Definition 4.1. If A and B are C^*-algebras, a *correspondence from A to B* (or simply an $A - B$ correspondence) is a pair $(\mathcal{E}, \phi)$ consisting of a Hilbert B-module $\mathcal{E}$ together with a (possibly degenerate) *-homomorphism $\phi : A \to \mathcal{L}(\mathcal{E})$.

If (A, α) and (B, β) are G-algebras, then a G-equivariant $A - B$ correspondence $(\mathcal{E}, \phi, \gamma)$ consists of an $A - B$ correspondence $(\mathcal{E}, \phi)$ together with a strongly continuous action $\gamma : G \to \mathrm{Aut}(\mathcal{E})$ such that

$$\langle \gamma_s(\xi) \,|\, \gamma_s(\eta) \rangle_B = \beta_s(\langle \xi \,|\, \eta \rangle_B), \quad \gamma_s(\xi \cdot b) = \gamma_s(\xi)\beta_s(b)$$
$$\text{and} \quad \gamma_s(\phi(a)\xi) = \phi(\alpha_s(a))\gamma_s(\xi)$$

for all $a \in A, b \in B, \xi, \eta \in \mathcal{E}$ and $s \in G$.

Definition 4.2. We say that two G-equivariant $A - B$ correspondences $(\mathcal{E}, \phi, \gamma)$ and $(\mathcal{E}', \phi', \gamma')$ are *isomorphic* if there exists an isomorphism $U : \mathcal{E} \to \mathcal{E}'$ which preserves all structures. We say that $(\mathcal{E}, \phi, \gamma)$ and $(\mathcal{E}', \phi', \gamma')$ are *equivalent* if there exists an isomorphism between $(\phi(A)\mathcal{E}, \phi, \gamma)$ and $(\phi'(A)\mathcal{E}, \phi', \gamma')$. In particular, every correspondence $(\mathcal{E}, \phi, \gamma)$ is equivalent to the nondegenerate correspondence $(\phi(A)\mathcal{E}, \phi, \gamma)$.

We should note at this point that $\phi(A)\mathcal{E} = \{\phi(a)\xi : a \in A, \xi \in \mathcal{E}\}$ will be a closed G-invariant Hilbert B-submodule of $\mathcal{E}$ – just apply Cohen's factorisation theorem to the closed submodule $\overline{\mathrm{span}}\, \phi(A)\mathcal{E}$ to see that this must coincide with $\phi(A)\mathcal{E}$.

Note that allowing general (*i.e.*, possibly degenerate) correspondences makes it more straightforward to view an arbitrary *-homomorphism $\phi : A \to B$ as a

correspondence: it will be represented by the correspondence (B, ϕ) (or (B, ϕ, β) in the equivariant setting) where B is regarded as a Hilbert B-module in the canonical way. Of course, by our definition of equivalence, $\phi : A \to B$ will also be represented by the correspondence $(\phi(A)B, \phi)$. Composition of an $(A, \alpha)-(B, \beta)$ correspondence $(\mathcal{E}, \phi, \gamma)$ with a $(B, \beta) - (D, \delta)$ correspondence $(\mathcal{F}, \psi, \tau)$ is given by the internal tensor product construction $(\mathcal{E} \otimes_\psi \mathcal{F}, \phi \otimes 1, \gamma \otimes \tau)$ (or just $(\mathcal{E} \otimes_\psi \mathcal{F}, \phi \otimes 1)$ in the non-equivariant case). The following lemma is folklore and we omit a proof.

Lemma 4.3. *The construction of internal tensor products is associative up to isomorphism of correspondences. Moreover, we have canonical isomorphisms*

$$\mathcal{E} \otimes_\psi \mathcal{F} \cong \mathcal{E} \otimes_\psi (\psi(B)\mathcal{F}) \quad \textit{and} \quad (\phi(A)\mathcal{E}) \otimes_\psi \mathcal{F} \cong \phi \otimes 1(A)(\mathcal{E} \otimes_\psi \mathcal{F}).$$

The above lemma together with [18, Theorem 2.8] allows the following definition:

Definition 4.4. There is a unique category $\mathfrak{Corr}(G)$, which we call the *G-equivariant correspondence category*, in which the objects are G-C^*-algebras and the morphisms between two objects (A, α) and (B, β) are equivalence classes of $(A, \alpha) - (B, \beta)$ correspondences $(\mathcal{E}, \phi, \gamma)$ with composition of morphisms given by internal tensor products. We write $\mathfrak{Corr}$ for the correspondence category of C^*-algebras without group actions (*i.e.*, where $G = \{e\}$ is the trivial group).

By our definition of equivalence of correspondences together with the above lemma, our category $\mathfrak{Corr}(G)$ is isomorphic to the category $\mathcal{A}(G)$ defined in [18, Theorem 2.8] in which the objects are C^*-algebras and the morphisms are isomorphism classes of *nondegenerate* G-equivariant correspondences. More precisely, the isomorphism $\mathfrak{Corr}(G) \xrightarrow{\sim} \mathcal{A}(G)$ is given by the assignment $[(\mathcal{E}, \phi, \gamma)] \to [(\phi(A)\mathcal{E}, \phi, \gamma)]$.

For every object (A, α) of $\mathfrak{Corr}(G)$, the identity morphism on (A, α) is given by the equivalence class of the correspondence $(A, \mathrm{id}_A, \alpha)$. It has been shown in [18, Lemma 2.4 and Remark 2.9] that the invertible morphisms in $\mathfrak{Corr}(G) \cong \mathcal{A}(G)$ are precisely the equivalence classes of equivariant Morita-equivalence bimodules (which we simply call *equivalence bimodules* below).

We now want to discuss under what conditions one can extend a crossed-product functor $(A, \alpha) \to A \rtimes_{\alpha,\mu} G$, which is functorial for G-equivariant *-homomorphisms to a functor from the correspondence category $\mathfrak{Corr}(G)$ to the correspondence category $\mathfrak{Corr} = \mathfrak{Corr}(\{e\})$. For this it is necessary to say what the functor should do on morphisms.

For this recall that if $(\mathcal{E}, \phi, \gamma)$ is a G-equivariant correspondence from (A, α) to (B, β), then there is a canonical construction of a "correspondence" $(C_c(G, \mathcal{E}), \phi \rtimes_c G)$ on the level of continuous functions with compact supports where the actions and inner products are given by

$$\langle x \,|\, y \rangle_{C_c(G,B)}(t) := \int_G \beta_{s^{-1}}(\langle x(s) \,|\, y(st) \rangle_B)\, ds$$
$$(x \cdot \varphi)(t) := \int_G x(s)\beta_s(\varphi(s^{-1}t))\, ds$$

for all $x, y \in C_c(G, \mathcal{E})$, $\varphi \in C_c(G, B)$, and

$$(\phi \rtimes_c G(f)x)(t) := \int_G \phi(f(s))\gamma_s(x(s^{-1}t))\, ds, \tag{4.5}$$

for all $x \in C_c(G, \mathcal{E})$ and $f \in C_c(G, A)$. It is well known (*e.g.*, see [11, 28]) that this construction completes to a correspondence $(\mathcal{E} \rtimes_{\gamma,\mathrm{u}} G, \phi \rtimes_{\mathrm{u}} G)$ between the maximal

crossed products $A \rtimes_{\alpha,\mathrm{u}} G$ and $B \rtimes_{\beta,\mathrm{u}} G$. We simply regard $\langle x \,|\, y\rangle_{C_c(G,B)}$ as an element of $B \rtimes_{\beta,\mathrm{u}} G$ and take completion with respect to the corresponding norm $\|x\|_{\mathrm{u}} := \sqrt{\|\langle x \,|\, x\rangle_{C_c(G,B)}\|_{\mathrm{u}}}$ on $C_c(G,\mathcal{E})$.

On the right hand side we can do a similar procedure in complete generality by regarding $\langle x \mid y\rangle_{C_c(G,B)}$ as an element in $B \rtimes_{\beta,\mu} G$ for any exotic crossed-product functor $\rtimes_\mu$: the completion $\mathcal{E} \rtimes_{\gamma,\mu} G$ of $C_c(G,\mathcal{E})$ with respect to $\|x\| := \sqrt{\|\langle x \,|\, x\rangle_{C_c(G,B)}\|_\mu}$ will always be a Hilbert $B \rtimes_{\beta,\mu} G$-module. Moreover, we have a canonical isomorphism of Hilbert $B \rtimes_{\beta,\mu} G$-modules:

$$(\mathcal{E} \otimes_{\gamma,\mathrm{u}} G) \otimes_{B \rtimes_{\beta,\mathrm{u}} G} (B \rtimes_{\beta,\mu} G) \xrightarrow{\sim} \mathcal{E} \rtimes_{\gamma,\mu} G, \tag{4.6}$$

sending $x \otimes b$ to $x \cdot b$, where the universal crossed product acts on the μ-crossed product via the quotient map.

The problem we need to address is the question whether the left action of $C_c(G,A)$ on $C_c(G,\mathcal{E})$ given by (4.5) will always extend to $A \rtimes_{\alpha,\mu} G$. We should also want that $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G \cong \mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G)$ if the left action exists. Indeed, we shall see below (Remark 4.19) that this fails for all BG-functors different from the universal crossed product, which is the BG-functor corresponding to the full group algebra $C^*(G)$. On the other hand, it has been already shown in [9, Section 2] that all KLQ-functors extend to the correspondence category. We shall give a short alternative argument for this at the end of this section.

We shall now introduce a few conditions that a functor may or may not have, which are related to these questions.

Definition 4.7. Let $(A,\alpha) \to A \rtimes_{\alpha,\mu} G$ be a crossed-product functor. Then we say

(1) the functor is *strongly Morita compatible* if for every G-equivariant $(A,\alpha) - (B,\beta)$ equivalence bimodule $(\mathcal{E},\gamma)$, the left action of $C_c(G,A)$ on $C_c(G,\mathcal{E})$ extends to an action of $A \rtimes_{\alpha,\mu} G$ on $\mathcal{E} \rtimes_{\gamma,\mu} G$ and makes $\mathcal{E} \rtimes_{\gamma,\mu} G$ an $A \rtimes_{\alpha,\mu} G - B \rtimes_{\beta,\mu} G$ equivalence bimodule.
(2) The functor has the *projection property* (resp. *full projection property*) if for every G-algebra A and every G-invariant (resp. full) projection $p \in \mathcal{M}(A)$, the inclusion $\iota : pAp \hookrightarrow A$ descends to a faithful homomorphism $\iota \rtimes_\mu G : pAp \rtimes_{\alpha,\mu} G \to A \rtimes_{\alpha,\mu} G$.
(3) The functor has the *hereditary-subalgebra property* if for every hereditary G-invariant subalgebra B of A, the inclusion $\iota : B \hookrightarrow A$ descends to a faithful map $\iota \rtimes_\mu G : B \rtimes_{\alpha,\mu} G \to A \rtimes_{\alpha,\mu} G$.
(4) The functor has the *cp map property* if for any completely positive and G-equivariant map $\phi : A \to B$ of G-algebras, the map

$$C_c(G,A) \to C_c(G,B),\ f \mapsto \phi \circ f$$

extends to a completely positive map from $A \rtimes_{\alpha,\mu} G$ to $B \rtimes_{\beta,\mu} G$.

Remark 4.8. The (full) projection property can be reformulated as follows: For every G-algebra A and for every G-invariant (full) projection $p \in \mathcal{M}(A)$, we have $pAp \rtimes_{\alpha,\mu} G = \tilde{p}(A \rtimes_{\alpha,\mu} G)\tilde{p}$, where $\tilde{p}$ denotes the canonical image of p in $\mathcal{M}(A \rtimes_{\alpha,\mu} G)$. This means that the closure of $C_c(G,pAp) = \tilde{p}C_c(G,A)\tilde{p}$ inside $A \rtimes_{\alpha,\mu} G$ coincides with $pAp \rtimes_{\alpha,\mu} G$.

Recall that the ideal property has been introduced in Definition 3.2. Since ideals are hereditary subalgebras it is weaker than the hereditary-subalgebra property.

Recall also from Lemma 3.3 that the ideal property is equivalent to the property that the functor extends to generalised homomorphisms $\phi : A \to \mathcal{M}(B)$.

Theorem 4.9. *Let $(A, \alpha) \to A \rtimes_{\alpha,\mu} G$ be a crossed-product functor. Then the following conditions are equivalent:*

1. *The functor extends to a correspondence functor from $\mathfrak{Corr}(G)$ to $\mathfrak{Corr}$ sending an equivalence class of a correspondence $(\mathcal{E}, \phi, \gamma)$ to the equivalence class of the correspondence $(\mathcal{E} \rtimes_{\gamma,\mu} G, \phi \rtimes_\mu G)$.*
2. *For every G-equivariant (right) Hilbert (B, β)-module $(\mathcal{E}, \gamma)$, the left action of $\mathcal{K}(\mathcal{E})$ on $\mathcal{E}$ descents to an isomorphism $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G \cong \mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G)$.*
3. *The functor is strongly Morita compatible and has the ideal property.*
4. *The functor has the hereditary subalgebra property.*
5. *The functor has the projection property.*
6. *The functor has the cp map property.*

If the above equivalent conditions hold, we say that $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ is a correspondence crossed-product functor*.*

We prepare the proof of Theorem 4.9 with some lemmas:

Lemma 4.10. *Suppose that the functor $(A, \alpha) \to A \rtimes_{\alpha,\mu} G$ extends to a correspondence functor from $\mathfrak{Corr}(G)$ to $\mathfrak{Corr}$. Then it has the ideal property.*

Proof. Let $\phi : A \to \mathcal{M}(B)$ be any G-equivariant generalised homomorphism from A to B. Consider the $(A, \alpha) - (B, \beta)$ correspondence (B, ϕ, β). It is clear that the μ-crossed product of B by G, if B is viewed as a Hilbert B-module, is equal to the C^*-algebra-crossed product $B \rtimes_\mu G$. Hence the left action of $A \rtimes_\mu G$ on the module exists if and only if our functor is functorial for generalised homomorphisms. By Lemma 3.3 this is equivalent to the ideal property. □

Lemma 4.11. *Let $(A, \alpha) \to A \rtimes_{\alpha,\mu} G$ be a crossed-product functor. Then the following are equivalent:*

1. *$\rtimes_\mu$ has the projection property.*
2. *$\rtimes_\mu$ has the full projection property and the ideal property.*

Proof. (2) $\Rightarrow$ (1) Let $p \in \mathcal{M}(A)$ be a G-invariant projection. Then $I := \overline{ApA}$ is a G-invariant ideal and p can be viewed as a full G-invariant projection in $\mathcal{M}(I)$. Then $pAp = pIp$ and the map $\iota \rtimes_\mu G : pAp \rtimes_\mu G \to A \rtimes_\mu G$ is the composition of the faithful inclusions

$$pAp \rtimes_\mu G \overset{\iota \rtimes_\mu G}{\hookrightarrow} I \rtimes_\mu G \overset{\iota \rtimes_\mu G}{\hookrightarrow} A \rtimes_\mu G.$$

For the converse direction, assume that $\rtimes_\mu$ has the projection property. Let I be a G-invariant ideal in the G-algebra A. Consider the algebra $L := \begin{pmatrix} I & I \\ I & A \end{pmatrix}$ with the obvious G-action. Then $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are opposite G-invariant projections in $\mathcal{M}(L)$ and by the projection property we see that p and q map to opposite projections $\tilde{p}$ and $\tilde{q}$ in $\mathcal{M}(L \rtimes_\mu G)$ such that $I \rtimes_\mu G = \tilde{p}(L \rtimes_\mu G)\tilde{p}$ and $A \rtimes_\mu G = \tilde{q}(L \rtimes_\mu G)\tilde{q}$. One easily checks that the right Hilbert $A \rtimes_\mu G$-module $\tilde{p}(L \rtimes_\mu G)\tilde{q}$ is the completion $I \rtimes_{\tilde{\mu}} G$ of $C_c(G, I) = C_c(G, pLq)$ with respect to the norm given by the inclusion $C_c(G, I) \hookrightarrow A \rtimes_\mu G$. On the other hand, regarding $\tilde{p}(L \rtimes_\mu G)\tilde{q}$ as a left Hilbert $I \rtimes_\mu G$-module in the canonical way, it will be the

completion of $C_c(G, I) = C_c(G, pLq)$ with respect to the norm on $I \rtimes_\mu G$. Hence $I \rtimes_{\tilde{\mu}} G = \tilde{p}(L \rtimes_\mu G)\tilde{q} = I \rtimes_\mu G$ and the result follows. □

The next lemma deals with various versions of Morita compatibility:

Lemma 4.12. *Let $\rtimes_\mu$ be a crossed-product functor for G. Then the following are equivalent:*

1. *For each full G-equivariant Hilbert (B, β)-module $(\mathcal{E}, \gamma)$ the left action of $\mathcal{K}(\mathcal{E})$ on $\mathcal{E}$ descents to an isomorphism $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G \cong \mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G)$.*
2. *$\rtimes_\mu$ is strongly Morita compatible.*
3. *$\rtimes_\mu$ has the hereditary subalgebra property for* full *G-invariant hereditary subalgebras $B \subseteq A$ (*i.e.*, $\overline{\operatorname{span}}\, ABA = A$).*
4. *$\rtimes_\mu$ has the full projection property.*

Proof. We prove (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2).

(1) $\Leftrightarrow$ (2) follows from the fact that if $(\mathcal{E}, \gamma)$ is a G-equivariant $(A, \alpha) - (B, \beta)$ equivalence bimodule, then the left action of A on $\mathcal{E}$ induces a G-equivariant isomorphism $(A, \alpha) \cong (\mathcal{K}(\mathcal{E}), \mathrm{Ad}\gamma)$ and, conversely, that every full G-equivariant Hilbert B-module is a G-equivariant $\mathcal{K}(\mathcal{E}) - B$ equivalence bimodule.

(2) $\Rightarrow$ (3): Recall that a closed subalgebra $B \subseteq A$ is a *full* hereditary subalgebra, if $BAB \subseteq B$ and $A = \overline{\operatorname{span}}\, ABA$. Then BA will be a $B - A$-equivalence bimodule. Let $B \rtimes_{\alpha,\tilde{\mu}} G := \iota \rtimes_\mu G(B \rtimes_{\beta,\mu} G)$; this coincides with the closure of $C_c(G, B)$ inside $A \rtimes_{\alpha,\mu} G$. Then a simple computation on the level of functions with compact supports shows that $B \rtimes_{\alpha,\tilde{\mu}} G$ is a full hereditary subalgebra of $A \rtimes_{\alpha,\mu} G$. For functions $f, g \in C_c(G, B) * C_c(G, A) \subseteq C_c(G, BA)$ it is clear that the $A \rtimes_{\alpha,\mu} G$-valued inner products given by viewing f, g as elements of $(B \rtimes_{\alpha,\tilde{\mu}} G)(A \rtimes_{\alpha,\mu} G)$ or as elements of the module $(BA) \rtimes_\mu G$ coincide. Thus we see that the inclusion $C_c(G, B) * C_c(G, A) \hookrightarrow C_c(G, BA)$ extends to an isomorphism $(B \rtimes_{\alpha,\tilde{\mu}} G)(A \rtimes_{\alpha,\mu} G) \cong (BA) \rtimes_\mu G$ of Hilbert $A \rtimes_{\alpha,\mu} G$-modules. But this induces an isomorphism of the compact operators

$$B \rtimes_{\alpha,\tilde{\mu}} G \cong \mathcal{K}((B \rtimes_{\beta,\tilde{\mu}} G)(A \rtimes_{\alpha,\mu} G)) \cong \mathcal{K}((BA) \rtimes_\mu G) \cong B \rtimes_{\alpha,\mu} G,$$

which extends the identity map on $C_c(G, B)$, where the last isomorphism follows from the strong Morita invariance of our functor.

(3) $\Rightarrow$ (4): follows from the fact that if $p \in \mathcal{M}(A)$ is a full G-invariant projection, then pAp is a full G-invariant hereditary subalgebra of A.

(4) $\Rightarrow$ (2): Assume that $(A, \alpha) \to A \rtimes_{\alpha,\mu} G$ satisfies the full projection property. Let $(\mathcal{E}, \gamma)$ be a $(A, \alpha) - (B, \beta)$ equivalence bimodule. Then we can form the linking algebra $L = \begin{pmatrix} A & \mathcal{E} \\ \mathcal{E}^* & B \end{pmatrix}$ equipped with the action $\sigma = \begin{pmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{pmatrix}$. Then $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are opposite full projections in $\mathcal{M}(L)$ and by the projection property we see that p and q map to opposite full projections $\tilde{p}$ and $\tilde{q}$ in $\mathcal{M}(L \rtimes_{\sigma,\mu} G)$ such that $\tilde{p}(L \rtimes_{\sigma,\mu} G)\tilde{q}$ is an equivalence bimodule between $A \rtimes_{\alpha,\mu} G = \tilde{p}(L \rtimes_{\sigma,\mu} G)\tilde{p}$ and $B \rtimes_{\beta,\mu} G = \tilde{q}(L \rtimes_{\sigma,\mu} G)\tilde{q}$. One easily checks that $C_c(G, \mathcal{E}) = C_c(G, pLq)$ equipped with the norm coming from the $B \rtimes_{\beta,\mu} G$-valued inner product embeds isometrically and densely into $\tilde{p}(L \rtimes_{\sigma,\mu} G)\tilde{q}$, hence we get $\mathcal{E} \rtimes_{\gamma,\mu} G = \tilde{q}(L \rtimes_{\sigma,\mu} G)\tilde{q}$. Therefore $\mathcal{E} \rtimes_{\gamma,\mu} G$ becomes an $A \rtimes_{\alpha,\mu} G - B \rtimes_{\beta,\mu} G$ equivalence bimodule. □

We are now ready for the

Proof of Theorem 4.9. We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Assume that $(A,\alpha) \mapsto A \rtimes_{\alpha,\mu} G$ extends to a correspondence functor. Then by Lemma 4.10 we know that the functor has the ideal property. Let $(\mathcal{E},\gamma)$ be a Hilbert (B,β)-module. Let $I = \overline{\mathrm{span}}\langle \mathcal{E}\,|\,\mathcal{E}\rangle_B$. Then the ideal property implies that viewing $(\mathcal{E},\gamma)$ as a Hilbert (I,β)-module or as a Hilbert (B,β)-module does not change the induced norm $\|\cdot\|_\mu$ on $C_c(G,\mathcal{E})$, and hence in both cases we get the same completion $\mathcal{E} \rtimes_{\gamma,\mu} G$. Thus we may assume without loss of generality that $\mathcal{E}$ is a full Hilbert B-module. But then the equivalence class of $(\mathcal{E},\gamma)$ will be an isomorphism from $(\mathcal{K}(\mathcal{E}), \mathrm{Ad}\gamma)$ to (B,β) in the equivariant correspondence category $\mathfrak{Corr}(G)$, which by the properties of a functor must be sent to an isomorphism from $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G$ to $B \rtimes_{\beta,\mu} G$ in $\mathfrak{Corr}$, which are equivalence classes of equivalence bimodules. Since the left action of $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G$ on $\mathcal{E} \rtimes_{\gamma,\mu} G$ is obviously nondegenerate, it follows that $\mathcal{E} \rtimes_{\gamma,\mu} G$ is a $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G - B \rtimes_{\beta,\mu} G$ equivalence bimodule. This implies the desired isomorphism.

(2) $\Rightarrow$ (3) It follows clearly from (2) applied to full Hilbert modules that the functor $(A,\alpha) \mapsto A \rtimes_{\alpha,\mu} G$ is strongly Morita compatible. To see that it has the ideal property, let I be a G-invariant closed ideal in A for some G-algebra (A,α). Regard (I,α) as a G-invariant Hilbert (A,α)-module in the canonical way. To avoid confusion, we write I_A for the Hilbert A-module I. We then have $(I,\alpha) = (\mathcal{K}(I_A), \mathrm{Ad}\gamma)$. Now if the descent $\iota \rtimes_\mu G : I \rtimes_\mu G \to A \rtimes_\mu G$ is not injective, it induces a new norm $\|\cdot\|_{\tilde{\mu}}$ on $C_c(G,I)$ which is (at least on some elements of $C_c(G,I)$) strictly smaller than the given norm $\|\cdot\|_\mu$. It is then easily checked that $I_A \rtimes_\mu G = I \rtimes_{\tilde{\mu}} G$. This also implies $\mathcal{K}(I_A \rtimes_\mu G) = I \rtimes_{\tilde{\mu}} G$ which will be a proper quotient of $\mathcal{K}(I_A) \rtimes_\mu G = I \rtimes_\mu G$, which contradicts (2).

(3) $\Rightarrow$ (4) Let $B \subseteq A$ be a G-invariant hereditary subalgebra of A. Then BA will be a $B - ABA$-equivalence bimodule, where ABA denotes the closed ideal of A generated by B. Since, by assumption, the μ-crossed-product functor satisfies the ideal property, we may assume without loss of generality that B is *full* in the sense that $ABA = A$. But then the result follows from part (2) $\Leftrightarrow$ (4) of Lemma 4.12.

(4) $\Rightarrow$ (5) This follows from the fact that corners are hereditary subalgebras.

(5) $\Rightarrow$ (6) Let $\phi : A \to B$ be a G-equivariant completely positive map between G-algebras; we may assume ϕ is non-zero. Multiplying ϕ by a suitable positive constant, we may assume that ϕ is contractive. It then follows from [7, Proposition 2.2.1] that the (unital) map defined on unitisations by

$$\widetilde{\phi} : \widetilde{A} \to \widetilde{B},\ \widetilde{\phi}(a + z1_A) = \phi(a) + z1_B$$

is also completely positive and contractive; it is moreover clearly equivariant for the extended actions of G.

Now, assume that $\widetilde{B}$ is represented faithfully, non-degenerately, and covariantly on a Hilbert space $\mathcal{H}$. Identify $\widetilde{B}$ with its image in $\mathcal{L}(\mathcal{H})$, and write u for the given representation of G on $\mathcal{H}$. Define a bilinear form on the algebraic tensor product $\widetilde{A} \odot \mathcal{H}$ by

$$\Big\langle \sum_{i=1}^n a_i \otimes \xi_i, \sum_{j=1}^m b_j \otimes \eta_j \Big\rangle := \sum_{i,j} \langle \xi_i, \widetilde{\phi}(a_i^* b_j)\eta_j\rangle.$$

As $\widetilde{\phi}$ is completely positive, this form is positive semi-definite, so separation and completion gives a Hilbert space $\mathcal{H}'$. If α is the action of G on $\widetilde{A}$, it follows from the fact that ϕ is equivariant that the formula

$$v_g : \sum_{i=1}^{n} a_i \otimes \xi_i \mapsto \sum_{i=1}^{n} \alpha_g(a_i) \otimes u_g \xi_i$$

defines a unitary action of G on $\mathcal{H}'$. Moreover, the action of $\widetilde{A}$ on $\widetilde{A} \odot \mathcal{H}$ defined by

$$\pi(a) \cdot \Big(\sum_{i=1}^{n} a_i \otimes \xi_i \Big) := \sum_{i=1}^{n} a a_i \otimes \xi_i$$

gives rise to a bounded representation of $\widetilde{A}$ on $\mathcal{H}'$ by the Cauchy-Schwarz inequality for ϕ. We also write π for the corresponding representation of $\widetilde{A}$ on $\mathcal{H}'$ and note that this is moreover covariant for v.

The formula

$$V : \mathcal{H} \to \mathcal{H}',\ \xi \mapsto 1_{\widetilde{A}} \otimes \xi$$

is easily seen to define an equivariant isometry with adjoint given by

$$V^* : \mathcal{H}' \to \mathcal{H},\ \sum_{i=1}^{n} a_i \otimes \xi_i \mapsto \sum_{i=1}^{n} \widetilde{\phi}(a_i)\xi_i.$$

Using these formulas, one sees that for all $a \in \widetilde{A}$ we have

$$\widetilde{\phi}(a) = V^* \pi(a) V \tag{4.13}$$

as elements of $\mathcal{L}(\mathcal{H})$.

To complete the proof, let C be the C^*-subalgebra of $\mathcal{L}(\mathcal{H}')$ generated by $\pi(\widetilde{A})$, $VV^*\pi(\widetilde{A})$, $\pi(\widetilde{A})VV^*$ and $V\widetilde{B}V^*$. Note that the action $\mathrm{Ad}v$ of G on C induced by the unitary representation v is norm continuous, so C is a G-algebra with the induced action. Moreover, the *-homomorphism

$$\pi : \widetilde{A} \to \mathcal{L}(\mathcal{H}')$$

is equivariant and has image in C, so gives rise to a *-homomorphism on crossed products

$$\pi \rtimes_\mu G : \widetilde{A} \rtimes_\mu G \to C \rtimes_\mu G \tag{4.14}$$

by functoriality of $\rtimes_\mu$. Note that by construction of C, $p := VV^*$ is in the multiplier algebra of C and $pCp = V\widetilde{B}V^*$, so we have an equivariant *-isomorphism

$$\kappa : pCp \to B,\ x \mapsto V^* x V. \tag{4.15}$$

Now, if we denote by $\tilde{p}$ the element of $\mathcal{M}(C \rtimes_\mu G)$ induced by p, then by the projection property we have a *-isomorphism

$$\tilde{p}(C \rtimes_\mu G)\tilde{p} \cong pCp \rtimes_\mu G \tag{4.16}$$

and so a *-isomorphism

$$\psi : \tilde{p}(C \rtimes_\mu G)\tilde{p} \to \widetilde{B} \rtimes_\mu G \tag{4.17}$$

defined as the map on crossed products induced by the composition of the isomorphism in line (4.16) and the *-isomorphism on crossed products induced by the isomorphism in line (4.15).

Now, consider the composition of maps

$$A \rtimes_\mu G \to \widetilde{A} \rtimes_\mu G \overset{\pi \rtimes_\mu G}{\to} C \rtimes_\mu G \overset{x \mapsto \tilde{p}x\tilde{p}}{\longrightarrow} \tilde{p}(C \rtimes_\mu G)\tilde{p} \overset{\psi}{\to} \widetilde{B} \rtimes_\mu G$$

where the first map is the *-homomorphism on crossed products induced from the equivariant inclusion $A \mapsto \widetilde{A}$, the second is as in line (4.14), the third is compression by $\tilde{p}$, and the fourth is ψ as in line (4.17). Each map appearing in the sequence is completely positive, whence the composition is completely positive. Checking the image of $C_c(G, A)$ using the formula in line (4.13) shows that the map agrees with the map $f \mapsto \phi \circ f$ from $A \rtimes_\mu G$ to D, where D is the image of $B \rtimes_\mu G$ under the canonical map of this C^*-algebra into $\widetilde{B} \rtimes_\mu G$. However, as $\rtimes_\mu$ has the projection property, it also has the ideal property, and so D is just a copy of $B \rtimes_\mu G$. This completes the proof.

(6) $\Rightarrow$ (5) Let $\rtimes_\mu$ be a cp-functorial crossed product for G. Let p be a G-invariant projection in the multiplier algebra of a G-algebra A, and consider the G-equivariant maps

$$pAp \to A \to pAp,$$

where the first map is the canonical inclusion, and the second map is compression by p; note that the first map is a *-homomorphism, the second is completely positive, and the composition of the two is the identity on pAp. Functoriality then gives *-homomorphisms on crossed products

$$pAp \rtimes_\mu G \to A \rtimes_\mu G \to pAp \rtimes_\mu G$$

whose composition is the identity; in particular, the first *-homomorphism is injective, which is the projection property.

(5) $\Rightarrow$ (1) Assume that $(A, \alpha) \to A \rtimes_{\alpha,\mu} G$ satisfies the projection property. We first show that this implies (2), *i.e.*, for any Hilbert (B, β)-module $(\mathcal{E}, \gamma)$ we get $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G \cong \mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G)$ in the canonical way. By Lemma 4.11 we know that $\rtimes_\mu$ satisfies the ideal property. Hence we may assume without loss of generality that $\mathcal{E}$ is a full Hilbert B-module. The result then follows from (4) $\Leftrightarrow$ (1) in Lemma 4.12.

To see that $(A, \alpha) \to A \rtimes_{\alpha,\mu} G$ extends to a correspondence functor we can now use the fact that by the ideal property we get functoriality for generalised homomorphisms. Hence for any G-equivariant correspondence $(\mathcal{E}, \phi, \gamma)$ from (A, α) to (B, β), the homomorphism $\phi : A \to \mathcal{L}(\mathcal{E}) \cong \mathcal{M}(\mathcal{K}(\mathcal{E}))$ descends to a *-homomorphism

$$A \rtimes_\mu G \to \mathcal{M}(\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G) \cong \mathcal{L}(\mathcal{E} \rtimes_{\gamma,\mu} G),$$

and therefore provides a correspondence $(\mathcal{E} \rtimes_{\gamma,\mu} G, \phi \rtimes_\mu G)$ from $A \rtimes_{\alpha,\mu} G$ to $B \rtimes_{\beta,\mu} G$. We already saw above that it preserves isomorphisms (*i.e.*, equivalence bimodules) and compatibility with compositions is proved on the level of functions with compact supports as in [18, Chapter 3]. □

We now give a few applications of Theorem 4.9. It is already shown in [9, Corollary 2.9] that all KLQ-functors extend to correspondences. We now give a very short argument for this:

Corollary 4.18. *Every KLQ-functor has the projection property and therefore extends to a functor from* $\mathfrak{Corr}(G)$ *to* $\mathfrak{Corr}$.

Proof. Let $v : C^*(G) \to C^*_v(G)$ be a quotient map which absorbs the universal representation u_G. Then the corollary follows from commutativity of the diagram

$$\begin{array}{ccc} pAp \rtimes_{v_{KLQ}} G & \xrightarrow{(\mathrm{id}_{pAp\rtimes G}\otimes v)\circ\widehat{\alpha}} & \mathcal{M}(pAp \rtimes_{\mathrm{u}} G \otimes C^*_v(G)) \\ {\scriptstyle \iota\rtimes_\mu G}\downarrow & & \downarrow{\scriptstyle \iota\rtimes_{\mathrm{u}} G\otimes \mathrm{id}_{C^*_v(G)}} \\ A \rtimes_{v_{KLQ}} G & \xrightarrow{(\mathrm{id}_{A\rtimes G}\otimes v)\circ\widehat{\alpha}} & \mathcal{M}(A \rtimes_{\mathrm{u}} G \otimes C^*_v(G)) \end{array}$$

together with the fact that the projection property holds for the universal crossed-product functor. □

Remark 4.19. On the other hand, the Brown-Guentner functors of Section 2.2 are almost never correspondence functors. In case of the BG-functor corresponding to the regular representation λ_G this is well illustrated in Example 2.2 above. In fact, it has been already observed in [4, Lemma A.6] that a BG-crossed product is Morita compatible in their sense (see §7 below) if and only if it coincides with the universal crossed product. Since every correspondence functor is also Morita compatible (see Proposition 7.10 below), we also see that the universal crossed-product functors are the only BG-functors which are also correspondence functors.

The next corollary shows that if we derive a functor from a correspondence functor via tensoring with a G-algebra (D, δ), then the new functor is also a correspondence functor. At the same time we extend the tensor-product construction of §2.4 to the case of non-unital G-algebras:

Corollary 4.20. *Suppose that $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ is a crossed-product functor which satisfies the ideal property and let (D, δ) be any G-algebra. Then there is a crossed-product functor $\rtimes_{\mu^\nu_D}$, also satisfying the ideal property, defined as*

$$A \rtimes_{\alpha,\mu^\nu_D} G := j_A \rtimes_\mu G(A \rtimes_{\alpha,\mu} G) \subseteq \mathcal{M}((A \otimes_\nu D) \rtimes_{\alpha\otimes\delta,\mu} G),$$

where $\otimes_\nu$ is either the minimal or the maximal tensor product of A with D and $j_A : A \to \mathcal{M}(A \otimes_\nu D)$ denotes the canonical inclusion. Moreover, if $\rtimes_\mu$ is a correspondence functor, then so is $\rtimes_{\mu^\nu_D}$.

Proof. Let $\mathcal{M}_D(A\otimes_\nu D)$ denote the closed subalgebra of $\mathcal{M}(A\otimes_\mu D)$ which consists of all elements m such that $m(1 \otimes_\nu D) \subseteq A \otimes_\nu D$. Then for any (possibly degenerate) *-homomorphism $\phi : A \to \mathcal{M}(B)$ the *-homomorphism $\phi \otimes \mathrm{id}_D : A \otimes_\nu D \to \mathcal{M}(B \otimes_\nu D)$ extends uniquely to $\mathcal{M}_D(A \otimes_\nu D)$ (*e.g.*, see [18, Proposition A.6]). Now let $\mathcal{M}_{D,c}(A \otimes_\nu D)$ denote the set of G-continuous elements in $\mathcal{M}_D(A \otimes_\nu D)$. Then it follows from the ideal property that $(A \otimes_\nu D) \rtimes_\mu G$ is an essential ideal in $\mathcal{M}_{D,c}(A \otimes_\nu D) \rtimes_\mu G$, which implies that there is a canonical faithful inclusion of $\mathcal{M}_{D,c}(A \otimes_\nu D) \rtimes_\mu G$ into $\mathcal{M}((A \otimes_\nu D) \rtimes_\mu G)$. Hence, we can alternatively define $A \rtimes_{\alpha,\mu^\nu_D} G$ as the image of $j_A \rtimes_\mu G$ inside $\mathcal{M}_{D,c}(A \otimes_\nu D) \rtimes_\mu G$. A similar argument applies if we replace $\mathcal{M}_{D,c}(A \otimes_\nu D)$ by the C^*-algebra $\mathcal{M}_c(A \otimes_\nu D)$ of G-continuous elements in $\mathcal{M}(A \otimes_\nu D)$.

Now let $\phi : A \to \mathcal{M}(B)$ be a G-equivariant *-homomorphism. Then the diagram

$$\begin{array}{ccc} A \rtimes_\mu G & \xrightarrow{\phi\rtimes_\mu G} & \mathcal{M}(B \rtimes_\mu G) \\ {\scriptstyle j_A\rtimes_\mu G}\downarrow & & \downarrow{\scriptstyle j_B\rtimes_\mu G} \\ \mathcal{M}_{D,c}(A \otimes_\nu D) \rtimes_\mu G & \xrightarrow[(\phi\otimes_\nu \mathrm{id}_D)\rtimes_\mu G]{} & \mathcal{M}_c(B \otimes_\nu D) \rtimes_\mu G \subset \mathcal{M}((B \otimes_\nu D) \rtimes_\mu G) \end{array}$$

commutes, since $(j_B \otimes_\nu \mathrm{id}_D) \circ \phi = (\phi \otimes_\nu \mathrm{id}_D) \circ j_A$. It follows that the bottom arrow restricts to a well-defined *-homomorphism from $A \rtimes_{\mu^\nu_D} G$ into $\mathcal{M}(B \rtimes_{\mu^\nu_D} G)$. It follows that $\rtimes_{\mu^\nu_D}$ is a functor for generalised homomorphisms. By Lemma 3.3 this also proves that the functor $\rtimes_{\mu^\nu_D}$ has the ideal property.

Assume now that $\rtimes_\mu$ is a correspondence functor. To see that this is then also true for $\rtimes_{\mu^\nu_D}$, let $p \in \mathcal{M}(A)$ be any G-invariant projection and let $\iota : pAp \to A$ be the inclusion map. Consider the commutative diagram

$$\begin{tikzcd}
pAp \rtimes_\mu G \arrow[r, "\iota \rtimes_\mu G"] \arrow[d, "j_{pAp} \rtimes_\mu G"'] & A \rtimes_\mu G \arrow[d, "j_A \rtimes_\mu G"] \\
\mathcal{M}_{D,c}(pAp \otimes_\nu D) \rtimes_\mu G \arrow[r, "(\iota \otimes_\nu \mathrm{id}_D) \rtimes_\mu G"'] & \mathcal{M}_c(A \otimes_\nu D) \rtimes_\mu G \subset \mathcal{M}((A \otimes_\nu D) \rtimes_\mu G)
\end{tikzcd}$$

Since $\rtimes_\mu$ satisfies the projection property, we have injectivity of

$$(\iota \otimes \mathrm{id}_D) \rtimes_\mu G : (pAp \otimes_\nu D) \rtimes_\mu G \hookrightarrow (A \otimes_\nu D) \rtimes_\mu G$$

which then extends to a unique injective *-homomorphism $\mathcal{M}_{D,c}(pAp \otimes_\nu D) \rtimes_\mu G \to \mathcal{M}((A \otimes_\nu D) \rtimes_\mu G)$. Thus the lower horizontal map in the diagram is injective. But then the commutativity of the diagram implies injectivity of $\iota \rtimes_{\mu^\nu_D} G : pAp \rtimes_{\mu^\nu_G} G \to A \rtimes_{\mu^\nu_G} G$, as desired. □

Remark 4.21. In [4, Lemma 5.4] it is shown that if $\rtimes_\mu$ is exact and D is unital, then $\rtimes_{\mu^{\max}_D}$ is also exact. We should note that for non-unital D this need not be true. To see an example, let G be any non-exact group, let $D = C_0(G)$ equipped with the translation action and let $\rtimes_\mu = \rtimes_{\mathrm{u}}$. Then

$$(A \otimes C_0(G)) \rtimes_{\mathrm{u}} G \cong (A \otimes C_0(G)) \rtimes_{\mathrm{r}} G \cong A \otimes \mathcal{K}(L^2(G))$$

and the map $j_A \rtimes_{\mathrm{u}} G : A \rtimes_{\mathrm{u}} G \to \mathcal{M}(A \otimes C_0(G)) \rtimes_{\mathrm{u}} G$ factors through the (faithful) map $j_A \rtimes_{\mathrm{r}} G : A \rtimes_{\mathrm{r}} G \to \mathcal{M}(A \otimes C_0(G)) \rtimes_{\mathrm{u}} G)$. Therefore we get $\rtimes_{\mathrm{u}^{\max}_{C_0(G)}} = \rtimes_{\mathrm{r}}$, which by the choice of G is not exact.

We conclude this section by showing that a correspondence crossed-product functor allows a nice description of $\mathcal{K}(\mathcal{E} \rtimes_\mu G, \mathcal{F} \rtimes_\mu G)$ in which $(\mathcal{E}, \gamma)$ and $(\mathcal{F}, \tau)$ are two G-equivariant Hilbert (B, β)-modules. This will be useful for our discussion of KK-theory in the next section. For this we first observe that $\mathcal{K}(\mathcal{E}, \mathcal{F})$ can be regarded as a $\mathcal{K}(\mathcal{F}) - \mathcal{K}(\mathcal{E})$ correspondence with respect to the canonical left action of $\mathcal{K}(\mathcal{F})$ on $\mathcal{K}(\mathcal{E}, \mathcal{F})$ given by composition of operators and with the $\mathcal{K}(\mathcal{E})$-valued inner product given by

$$\langle T \,|\, S \rangle_{\mathcal{K}(\mathcal{E})} = T^* \circ S.$$

The G-actions γ and ν induce an action $\mathrm{Ad}(\gamma, \tau)$ of G on $\mathcal{K}(\mathcal{E}, \mathcal{F})$ by

$$\mathrm{Ad}(\gamma, \tau)_s(T) = \tau_{s^{-1}} T \gamma_s.$$

Then $(\mathcal{K}(\mathcal{E}, \mathcal{F}), \mathrm{Ad}(\gamma, \tau))$ becomes a G-equivariant $(\mathcal{K}(\mathcal{E}), \mathrm{Ad}\gamma) - (\mathcal{K}(\mathcal{F}), \mathrm{Ad}\tau)$ correspondence. If $\rtimes_\mu$ is any crossed-product functor for G, we may consider the crossed product $\mathcal{K}(\mathcal{E}, \mathcal{F}) \rtimes_{\mathrm{Ad}(\gamma,\tau),\mu} G$ as a completion of $C_c(G, \mathcal{K}(\mathcal{E}, \mathcal{F}))$ as described at the beginning of this section.

Lemma 4.22. *Suppose that $\rtimes_\mu$ is a correspondence crossed-product functor. Then there is a canonical isomorphism*

$$\mathcal{K}(\mathcal{E}, \mathcal{F}) \rtimes_{\mathrm{Ad}(\gamma,\tau),\mu} G \cong \mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G, \mathcal{F} \rtimes_{\tau,\mu} G)$$

which sends a function $f \in C_c(G, \mathcal{K}(\mathcal{E}, \mathcal{F})) \subseteq \mathcal{K}(\mathcal{E}, \mathcal{F}) \rtimes_\mu G$ *to the operator* $T_f \in \mathcal{K}(\mathcal{E} \rtimes_\mu G, \mathcal{F} \rtimes_\mu G)$ *given on the dense submodule* $C_c(G, \mathcal{E}) \subseteq \mathcal{E} \rtimes_{\gamma,\mu} G$ *by the convolution formula*

$$(T_f \xi)(s) = \int_G f(t)\gamma_s(\xi(t^{-1}s))\, dt.$$

Proof. Let $\mathcal{E} \oplus \mathcal{F}$ denote the direct sum of the Hilbert B-modules $\mathcal{E}$ and $\mathcal{F}$. Let $p, q \in \mathcal{L}(\mathcal{E} \oplus \mathcal{F})$ denote the orthogonal projections to $\mathcal{E}$ and $\mathcal{F}$. This gives a canonical decomposition

$$\mathcal{K}(\mathcal{E} \oplus \mathcal{F}) \cong \begin{pmatrix} \mathcal{K}(\mathcal{E}) & \mathcal{K}(\mathcal{F}, \mathcal{E}) \\ \mathcal{K}(\mathcal{E}, \mathcal{F}) & \mathcal{K}(\mathcal{F}) \end{pmatrix}$$

by identifying $\mathcal{K}(\mathcal{E}) \cong p\mathcal{K}(\mathcal{E} \oplus \mathcal{F})p$, $\mathcal{K}(\mathcal{F}, \mathcal{E}) = p\mathcal{K}(\mathcal{E} \oplus \mathcal{F})q$ and so on. The projections p and q are G-invariant and therefore map to opposite G-invariant projections $\tilde{p}$ and $\tilde{q}$ in $\mathcal{M}(\mathcal{K}(\mathcal{E} \oplus \mathcal{F}) \rtimes_{\mathrm{Ad}(\gamma\oplus\tau),\mu} G)$ under the canonical map. Taking crossed products it follows from the properties of a correspondence functor shown in Theorem 4.9 that we get a decomposition

$$\begin{aligned} \mathcal{K}\big((\mathcal{E} \oplus \mathcal{F}) \rtimes_{\gamma\oplus\tau,\mu} G\big) &\cong \mathcal{K}(\mathcal{E} \oplus \mathcal{F}) \rtimes_{\mathrm{Ad}(\gamma\oplus\tau),\mu} G \\ &= \begin{pmatrix} \mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G & \tilde{p}\big(\mathcal{K}(\mathcal{F} \oplus \mathcal{E}) \rtimes_\mu G\big)\tilde{q} \\ \tilde{q}\big(\mathcal{K}(\mathcal{F} \oplus \mathcal{E}) \rtimes_\mu G\big)\tilde{p} & \mathcal{K}(\mathcal{F}) \rtimes_{\mathrm{Ad}\tau,\mu} G \end{pmatrix} \end{aligned}$$

Notice that we have the identity

$$(\mathcal{E} \oplus \mathcal{F}) \rtimes_{\gamma\oplus\tau,\mu} G \cong \mathcal{E} \rtimes_{\gamma,\mu} G \oplus \mathcal{F} \rtimes_{\tau,\mu} G,$$

which follows directly from the definition of the inner products on $C_c(G, \mathcal{E} \oplus \mathcal{F})$. It implies the decomposition

$$\begin{aligned} \mathcal{K}((\mathcal{E} \oplus F) \rtimes_{\gamma\oplus\tau,\mu} G) &\cong \mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G \oplus \mathcal{F} \rtimes_{\tau,\mu} G) \\ &\cong \begin{pmatrix} \mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G) & \mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G, \mathcal{F} \rtimes_{\tau,\mu} G) \\ \mathcal{K}(\mathcal{F} \rtimes_{\tau,\mu} G, \mathcal{E} \rtimes_{\gamma,\mu} G) & \mathcal{K}(\mathcal{F} \rtimes_{\tau,\mu} G) \end{pmatrix} \\ &\cong \begin{pmatrix} \mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G & \mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G, \mathcal{F} \rtimes_{\tau,\mu} G) \\ \mathcal{K}(\mathcal{F} \rtimes_{\tau,\mu} G, \mathcal{E} \rtimes_{\gamma,\mu} G) & \mathcal{K}(\mathcal{F}) \rtimes_{\mathrm{Ad}\tau,\mu} G \end{pmatrix}. \end{aligned}$$

Comparing both isomorphisms on functions in $C_c(G, \mathcal{K}(\mathcal{E} \oplus \mathcal{F}))$, we see that they agree and that the isomorphism of the upper right corner is given on the level of functions in $C_c(G, \mathcal{K}(\mathcal{E}, \mathcal{F})) = \tilde{p}C_c(G, \mathcal{K}(\mathcal{E} \oplus \mathcal{F}))\tilde{q}$ as in the statement. □

5. KK-descent and the Baum-Connes conjecture

In this section we want to show that every crossed-product functor $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ which extends to a functor $\mathfrak{Corr}(G) \to \mathfrak{Corr}$ as discussed in the previous section will always allow a descent in Kasparov's bivariant KK-theory. Let us briefly recall the cycles in Kasparov's equivariant KK-theory group $KK^G(A, B)$ in which (A, α) and (B, β) are G-algebras. As in [28] we allow $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebras and Hilbert modules. In this case all G-actions have to commute with the given gradings, so the grading on the objects (A, α) and morphisms $(\mathcal{E}, \gamma)$ will always induce canonical gradings on their crossed products $A \rtimes_{\alpha,\mu} G$ and $\mathcal{E} \rtimes_{\gamma,\mu} G$, respectively. For convenience and to make sure that Kasparov products always exist we shall assume that A and B are separable and that G is second countable.

If (A, α) and (B, β) are two G-algebras, we define $\mathbb{E}^G(A, B)$ to be the set of all quadruples $(\mathcal{E}, \gamma, \phi, T)$ such that $(\mathcal{E}, \phi, \gamma)$ is a correspondence from (A, α) to (B, β)

such that $\mathcal{E}$ is countably generated, and $T \in \mathcal{L}(\mathcal{E})$ is an operator of degree one satisfying the following conditions:

(1) $s \mapsto \phi(\alpha_s(a))T$ is continuous for all $a \in A$, and
(2) $[T, \phi(a)], (T - T^*)\phi(a), (T^2 - 1)\phi(a), (\mathrm{Ad}\gamma_s(T) - T)\phi(a) \in \mathcal{K}(\mathcal{E})$

for all $a \in A$, where we use the graded commutator $[\cdot, \cdot]$ in (2). Two Kasparov cycles $(\mathcal{E}_1, \gamma_1, \phi_1, T_1)$ and $(\mathcal{E}_2, \gamma_2, \phi_2, T_2)$ are *isomorphic* if there exists a bijection $u : \mathcal{E}_1 \to \mathcal{E}_2$ which preserves all structures. If G is trivial, there is no action γ on $\mathcal{E}$ and the elements in $\mathbb{E}(A, B)$ are just given by triples $(\mathcal{E}, \phi, T)$ with the obvious properties.

Write $B[0, 1]$ for the algebra of continuous functions from the unit interval $[0, 1]$ to B and $\epsilon_t : B[0, 1] \to B$ for evaluation at $t \in [0, 1]$. The action of G on $B[0, 1] = B \otimes C([0, 1])$ is given by $\beta \otimes \mathrm{id}$. A *homotopy* between two elements $(\mathcal{E}_0, \gamma_0, \phi_0, T_0)$ and $(\mathcal{E}_1, \gamma_1, \phi_1, T_1)$ in $\mathbb{E}^G(A, B)$ is an element $(\mathcal{E}, \gamma, \phi, T) \in \mathbb{E}^G(A, B[0, 1])$ such that

$$(\mathcal{E}_i, \gamma_i, \phi_i, T_i) \cong (\mathcal{E}\hat{\otimes}_{\varepsilon_i} B, \gamma\hat{\otimes}1, \phi\hat{\otimes}1, T\hat{\otimes}1), \quad i = 0, 1,$$

where, for each $t \in [0, 1]$, $\mathcal{E}\hat{\otimes}_{\varepsilon_t} B$ denotes the balanced tensor product $\mathcal{E} \otimes_{B[0,1]} B$ in which $B[0, 1]$ acts on B via ϵ_t. Then Kasparov defines $KK^G(A, B)$ as the set of homotopy classes in $\mathbb{E}^G(A, B)$. It is an abelian group with respect to addition given by taking direct sums of elements in $\mathbb{E}^G(A, B)$.

Recall also that for $i = 0, 1$ one may define $KK_0^G(A, B) := KK^G(A, B)$ and $KK_1^G(A, B) := KK^G(C_0(\mathbb{R}) \otimes A, B) \cong KK^G(A, C_0(\mathbb{R}) \otimes B)$ where G acts trivially on $C_0(\mathbb{R})$ (the last isomorphism follows from Kasparov's Bott-periodicity theorem for KK-theory).

Note that if $\mathcal{E}$ is a graded module, then there are natural gradings on the crossed products determined by applying the given gradings pointwise to functions with compact supports. The balanced tensor product of two graded modules carries the diagonal grading. The following proposition extends the descent given by Kasparov in [28] for full and reduced crossed products to arbitrary correspondence crossed-product functors.

Proposition 5.1. *Let $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ be a correspondence crossed-product functor. Then there is a well-defined descent homomorphism*

$$J_G^\mu : KK_i^G(A, B) \to KK_i(A \rtimes_{\alpha,\mu} G, B \rtimes_{\beta,\mu} G), \quad i = 0, 1,$$

given by sending a class $[\mathcal{E}, \gamma, \phi, T] \in KK_i^G(A, B)$ to the class

$$[\mathcal{E} \rtimes_{\gamma,\mu} G, \phi \rtimes_\mu G, T_\mu] \in KK_i(A \rtimes_{\alpha,\mu} G, B \rtimes_{\beta,\mu} G),$$

where $T_\mu \in \mathcal{L}(\mathcal{E} \rtimes_{\gamma,\mu} G)$ is the unique continuous extension of the operator $\tilde{T} : C_c(G, \mathcal{E}) \to C_c(G, \mathcal{E})$ given by $(\tilde{T}\xi)(s) := T(\xi(s))$ for all $s \in G$. The descent is compatible with Kasparov products, i.e., *we have*

$$J_G^\mu(x \otimes_B y) = J_G^\mu(x) \otimes_{B \rtimes_{\beta,\mu} G} J_G^\mu(y)$$

for all $x \in KK_i^G(A, B)$ and $y \in KK_j^G(B, D)$ and $J_G^\mu(1_B) = 1_{B \rtimes_{\beta,\mu} G}$.

Proof. We basically follow the proof of [28, Theorem on p. 172]. First observe that since $\mathcal{L}(\mathcal{E} \rtimes_{\gamma,\mu} G) = \mathcal{M}(\mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G)) = \mathcal{M}(\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G)$, the operator T_μ is just the image of T under the extension of the canonical inclusion from $\mathcal{K}(\mathcal{E})$ into $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G$ to $\mathcal{M}(\mathcal{K}(\mathcal{E})) \cong \mathcal{L}(\mathcal{E})$. It therefore exists.

Next, the conditions for $(\mathcal{E} \rtimes_{\gamma,\mu} G, \phi \rtimes_\mu G, T_\mu)$ follow from the fact that for any $a \in C_c(G, A)$ the elements $[\tilde{T}, \phi(a)], (\tilde{T}^2 - 1)\phi(a), (\tilde{T}^* - \tilde{T})\phi(a)$ (as operators on

$C_c(G,\mathcal{E}))$ lie in $C_c(G,\mathcal{K}(\mathcal{E}))$, and hence in $\mathcal{K}(\mathcal{E}) \rtimes_{\mathrm{Ad}\gamma,\mu} G \cong \mathcal{K}(\mathcal{E} \rtimes_{\gamma,\mu} G)$. This shows that $(\mathcal{E} \rtimes_{\beta,\mu} G, \phi \rtimes_\mu G, T_\mu)$ determines a class in $KK(A \rtimes_{\alpha,\mu} G, B \rtimes_{\beta,\mu} G)$.

To see that this class does not depend on the choice of the representative $(\mathcal{E},\gamma,\phi,T) \in \mathbb{E}^G(A,B)$ we need to observe that the procedure preserves homotopy. So assume now that $(\mathcal{E},\gamma,\phi,T) \in \mathbb{E}^G(A,B[0,1])$. By Lemma 3.6 we know that $B[0,1] \rtimes_{\beta\otimes\mathrm{id},\mu} G \cong (B \rtimes_{\beta,\mu} G)[0,1]$, hence $(\mathcal{E} \rtimes_{\gamma,\mu} G, \phi \rtimes_\mu G, T_\mu) \in \mathbb{E}(A \rtimes_{\alpha,\mu} G, B \rtimes_{\beta,\mu} G[0,1])$. It follows then from correspondence functoriality of our crossed-product functor that evaluation of $(\mathcal{E} \rtimes_{\gamma,\mu} G, \phi \rtimes_\mu G, T_\mu)$ at any $t \in [0,1]$ coincides with the descent of the evaluation of $(\mathcal{E},\gamma,\phi,T)$ at t (this implies that the modules coincide, but a short look at the operator on functions with compact supports also shows that the operators coincide). This shows that the descent gives a well-defined homomorphism of KK-groups. Using the isomorphism $(C_0(\mathbb{R}) \otimes A) \rtimes_\mu G \cong C_0(\mathbb{R}) \otimes (A \rtimes_\mu G)$ if G acts trivially on $\mathbb{R}$, which follows from Lemma 3.6, it follows that it preserves the dimension of the KK-groups.

Finally the fact that the descent is compatible with Kasparov products follows from the same arguments as used in the original proof of Kasparov as given on [28, page 173] for the full and reduced crossed products together with an application of Lemma 4.22 to $\mathcal{K}(\mathcal{E}_2, \mathcal{E}_1 \hat{\otimes}_B \mathcal{E}_2)$. □

Suppose now that (A,G,α) is a C^*-dynamical system with A separable and G second countable. Suppose that $A \rtimes_{\alpha,\nu} G$ is any quotient of the universal crossed product $A \rtimes_{\alpha,\mathrm{u}} G$ with quotient map $q_\nu : A \rtimes_{\alpha,\mathrm{u}} G \to A \rtimes_{\alpha,\nu} G$. Using the Baum-Connes assembly map

$$\mathrm{as}^{\mathrm{u}}_{(A,G)} : K^{\mathrm{top}}_*(G;A) \to K_*(A \rtimes_{\alpha,\mathrm{u}} G)$$

for the full crossed product, we obtain an assembly map

$$\mathrm{as}^{\nu}_{(A,G)} : K^{\mathrm{top}}_*(G;A) \to K_*(A \rtimes_{\alpha,\nu} G)$$

by putting

$$\mathrm{as}^{\nu}_{(A,G)} = q_{\nu,*} \circ \mathrm{as}^{\mathrm{u}}_{(A,G)} \,. \tag{5.2}$$

Of course, in general we cannot assume that this map has any good properties, since we should assume that there is a huge variety of quotients of $A \rtimes_{\alpha,\mathrm{u}} G$ with different K-theory groups. If the C^*-algebra $A \rtimes_{\alpha,\nu} G$ lies between the maximal and reduced crossed products, then the Baum-Connes conjecture predicts that $\mathrm{as}^{\nu}_{(A,G)}$ is split injective, but we cannot say more than that.

On the other hand, if G is K-amenable, then the quotient map $q_{\mathrm{r}} : A \rtimes_{\alpha,\mathrm{u}} G \to A \rtimes_{\alpha,\mathrm{r}} G$ induces an isomorphism of K-theory groups, and there might be a chance that a similar result will hold for intermediate crossed products. For arbitrary intermediate crossed products this is not true, as follows from [4, Example 6.4], where an easy counter example is given that applies to any K-amenable[1], non-amenable group. However, we shall see below that it holds for all correspondence crossed-product functors for any K-amenable group!

We need to recall the construction of the assembly map: For this assume that G is a second countable locally compact group and that $\underline{EG}$ is a universal proper G-space, *i.e.*, $\underline{EG}$ is a locally compact proper G-space such that for every locally compact

[1]The discussion in [4, Example 6.4] is for a-T-menable, non-amenable groups, but the same argument applies if one assumes K-amenability in place of a-T-menability.

G-space Y there is a (unique up to G-homotopy) continuous G-map $\varphi : Y \to \underline{EG}$. The topological K-theory of G with coefficient A is defined as

$$K_*^{\mathrm{top}}(G;A) := \lim_{X \subseteq \underline{EG}} KK_*^G(C_0(X), A),$$

where $X \subseteq \underline{EG}$ runs through all G-compact (which means that X is a closed G-invariant subset with $G\backslash X$ compact) subsets of $\underline{EG}$.

Now, using properness, for any such G-compact subset $X \subseteq \underline{EG}$ we may choose a cut-off function $c \in C_c(G)^+$ such that $\int_G c^2(s^{-1}x)\,ds = 1$ for all $x \in X$. Define $p_X(s,x) = \Delta_G(s)^{-1/2}c(x)x(s^{-1}x)$. It follows from the properties of c that the function $p_X \in C_c(G, C_0(X)) \subseteq C_0(X) \rtimes G$ is an orthogonal projection. As any two cut-off functions with the properties above are homotopic, the K-theory class $[p_X] \in K_*(C_0(X) \rtimes G)$ does not depend on the choice of the cut-off function c. Note also that proper actions on spaces are always amenable, hence the universal and reduced crossed products coincide. This also implies that all intermediate crossed products coincide. Using this and the above defined descent in KK-theory, we may now construct a direct assembly map for any correspondence crossed-product functor $(A,\alpha) \mapsto A \rtimes_{\alpha,\mu} G$ and separable G-algebra A given by the composition

$$\text{J-as}_X^{\mu} : KK_*^G(C_0(X), A) \xrightarrow{J^{\mu}} KK_*(C_0(X) \rtimes G, A \rtimes_{\alpha,\mu} G) \xrightarrow{[p_X]\otimes\cdot} K_*(A \rtimes_{\alpha,\mu} G).$$

Note that this is precisely the same construction as given by Baum, Connes and Higson in [3] for the universal or reduced crossed products, and as in those cases it is easy to check that the maps are compatible with the limit structure and therefore define a direct assembly map

$$\text{J-as}_{(A,G)}^{\mu} : K_*^{\mathrm{top}}(G;A) \to K_*(A \rtimes_{\alpha,\mu} G); \tag{5.3}$$

the notation J-as is meant to recall that we used the descent functor in the definition. However, the next result shows that this assembly map is the same as that in line (5.2) above: it is a KK-theoretic version of [4, Proposition 4.5].

Lemma 5.4. *Assume (A,G,α) is a C^*-dynamical system with A separable and G second countable. Assume $\rtimes_\mu$ is a correspondence crossed-product functor for G. Then the diagram*

$$\begin{tikzcd}
 & K_*(A \rtimes_{\mathrm{u}} G) \arrow[d, "(q_\mu)_*"] \\
K_*^{\mathrm{top}}(G;A) \arrow[ur, "\mathrm{as}^{\mathrm{u}}_{(A,G)}"] \arrow[r, "\text{J-as}^{\mu}_{(A,G)}"] \arrow[dr, "\mathrm{as}^{\mathrm{r}}_{(A,G)}"'] & K_*(A \rtimes_{\mu} G) \arrow[d, "(q^{\mu}_{\mathrm{r}})_*"] \\
 & K_*(A \rtimes_{\mathrm{r}} G)
\end{tikzcd}$$

commutes. In particular, the assembly maps $\mathrm{as}^{\mu}_{(A,G)}$ and $\text{J-as}^{\mu}_{(A,G)}$ of lines (5.2) and (5.3) above agree.

Proof. Let $[q_\mu] \in KK(A \rtimes_{\mathrm{u}} G, A \rtimes_\mu G)$ be the KK-class induced by the quotient *-homomorphism $q_\mu : A \rtimes_{\mathrm{u}} G \twoheadrightarrow A \rtimes_\mu G$. For commutativity of the top triangle, it

suffices to show that for any G-compact subset X of $\underline{EG}$ the diagram

$$\begin{tikzcd}
KK_*^G(C_0(X), A) \arrow[r, "J^{\mathrm{u}}"] \arrow[dd, "J^{\mu}"] & KK_*(C_0(X) \rtimes G, A \rtimes_{\alpha,\mathrm{u}} G) \arrow[d, "{[p_X]\otimes\cdot}"] \\
 & KK_*(\mathbb{C}, A \rtimes_{\alpha,\mathrm{u}} G) \arrow[d, "{\cdot\otimes[q_\mu]}"] \\
KK_*(C_0(X) \rtimes G, A \rtimes_{\alpha,\mu} G) \arrow[r, "{[p_X]\otimes\cdot}"] & KK_*(\mathbb{C}, A \rtimes_{\alpha,\mu} G)
\end{tikzcd}$$

commutes (we have used that $C_0(X) \rtimes G = C_0(X) \rtimes_\mu G$ to identify the group in the bottom left corner). In symbols, commutativity of this diagram means that for any $x \in KK_*^G(C_0(X), A)$ we have

$$([p_X] \otimes J^{\mathrm{u}}(x)) \otimes [q_\mu] = [p_X] \otimes J^\mu(x).$$

Using associativity of the Kasparov product, it thus suffices to show that for any $x \in KK_*^G(C_0(X), A)$, $J^{\mathrm{u}}(x) \otimes [q_\mu] = J^\mu(x)$.

Indeed, say x is represented by the quadruple $(\mathcal{E}, \gamma, \phi, T)$. Then $J^{\mathrm{u}}(x) \otimes [q_\mu]$ and $J^\mu(x)$ are represented by the triples

$$(\mathcal{E} \rtimes_{\gamma,\mathrm{u}} G \otimes_{B\rtimes_\beta G} B \rtimes_{\beta,\mu} G, \phi \rtimes_\mu G \otimes 1, T_{\mathrm{u}} \otimes 1) \text{ and } (\mathcal{E} \rtimes_{\gamma,\mu} G, \phi \rtimes_\mu G, T_\mu)$$

respectively. It is easy to see that the canonical isomorphism $\mathcal{E} \rtimes_{\gamma,\mathrm{u}} G \otimes_{B\rtimes_{\beta,\mathrm{u}}G} B \rtimes_{\beta,\mu} G \xrightarrow{\sim} \mathcal{E} \rtimes_{\gamma,\mu} G$ of Equation (4.6) sends $\phi \rtimes_\mu G \otimes 1$ to $\phi \rtimes_\mu G$ and $T_{\mathrm{u}} \otimes 1$ to T_μ, and hence induces an isomorphism of these triples.

This shows commutativity of the upper diagram. Commutativity of the lower one follows by similar arguments. □

Recall now that a group G is said to satisfy the *strong Baum-Connes conjecture* (or that G has a γ-element equal to one), if there exists an $\underline{EG} \rtimes G$-algebra $\mathcal{B}$ (which means that there is a G-equivariant nondegenerate *-homomorphism $C_0(\underline{EG})$ into the center $Z\mathcal{M}(\mathcal{B})$ of the multiplier algebra $\mathcal{M}(\mathcal{B})$), together with classes $D \in KK_*^G(\mathcal{B}, \mathbb{C})$ and $\eta \in KK_*^G(\mathbb{C}, \mathcal{B})$ such that $\eta \otimes_{\mathcal{B}} D = 1_G \in KK^G(\mathbb{C}, \mathbb{C})$. It was shown by Higson and Kasparov in [22] that all a-T-menable groups satisfy the strong Baum-Connes conjecture. We then get

Theorem 5.5. *Suppose that G is a locally compact group which satisfies the strong Baum-Connes conjecture and let $(A, \alpha) \to A \rtimes_{\alpha,\mu} G$ be a correspondence crossed-product functor. Then the μ-assembly map*

$$\mathrm{as}^\mu_{(A,G)} : K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_{\alpha,\mu} G)$$

is an isomorphism for all G-algebras (A, α).

This can be proved directly in the same way as the special case dealing with the reduced crossed product. However, we shall obtain the result as a consequence of the known isomorphism of the assembly map for the reduced crossed products (*e.g.*, see [36, Theorem 2.2] or [22, Theorem 1.1]) together with Lemma 5.4 above and Theorem 5.6 below, which in particular implies that for every K-amenable group G the quotient map $A \rtimes_{\mathrm{u}} G \twoheadrightarrow A \rtimes_\mu G$ induces an isomorphism in K-theory.

Recall that a locally compact group G is called *K-amenable* in the sense of Cuntz and Julg-Valette ([14, 25]) if the unit element $1_G \in KK^G(\mathbb{C}, \mathbb{C})$ can be represented by a cycle $(\mathcal{H}, \gamma, T)$ in which the unitary representation $\gamma : G \to \mathcal{U}(\mathcal{H})$ on the

Hilbert space $\mathcal{H}$ is weakly contained in the regular representation. In other words, its integrated form factors through a $*$-representation $\gamma : C_r^*(G) \to \mathcal{L}(\mathcal{H})$. (The left action of $\mathbb{C}$ on $\mathcal{H}$ is assumed to be by scalar multiplication, so we omit it in our notation.) It is shown by Tu in [36, Theorem 2.2] that every locally compact group which satisfies the strong Baum-Connes conjecture is also K-amenable and it is shown by Julg and Valette in [25, Proposition 3.4] that for any K-amenable group the regular representation

$$A \rtimes_{\alpha,\mathrm{u}} G \twoheadrightarrow A \rtimes_{\alpha,\mathrm{r}} G$$

induces a KK-equivalence for any G-C^*-algebra (A, α). We shall now extend this result to correspondence functors:

Theorem 5.6 (*cf.* [25, Proposition 3.4])**.** *Suppose that G is K-amenable and let $\rtimes_\mu$ be any correspondence functor for G. Then both maps in the sequence*

$$A \rtimes_{\alpha,\mathrm{u}} G \overset{q_\mu}{\twoheadrightarrow} A \rtimes_{\alpha,\mu} G \overset{q_{\mathrm{r}}^\mu}{\twoheadrightarrow} A \rtimes_{\alpha,\mathrm{r}} G$$

are KK-equivalences.

For the proof we need the following lemma. Its analogue for full crossed products was used in the proof of [25, Proposition 3.4].

Lemma 5.7. *Suppose that $\gamma : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation and (A, α) is a G-C^*-algebra. Consider $(\mathcal{H} \otimes A, \gamma \otimes \alpha)$ as a G-equivariant $A - A$ correspondence with the obvious $A = \mathbb{C} \otimes A$-valued inner product and left action $1_{\mathcal{H}} \otimes \mathrm{id}_A$ of A. Then, if $\rtimes_\mu$ is a correspondence functor for G, there is an isomorphism of $A \rtimes_\mu G - A \rtimes_\mu G$ correspondences*

$$(\mathcal{H} \otimes A) \rtimes_{\gamma\otimes\alpha,\mu} G \cong \mathcal{H} \otimes (A \rtimes_{\alpha,\mu} G)$$

where the left action of $A \rtimes_\mu G$ on $\mathcal{H} \otimes (A \rtimes_{\alpha,\mu} G)$ is given via the covariant homomorphism $(1_H \otimes i_A^\mu) \rtimes (\gamma \otimes i_G^\mu)$, where $(i_A^\mu, i_G^\mu) : (A, G) \to \mathcal{M}(A \rtimes_\mu G)$ denote the canonical maps.

Proof. We first note that for every G-equivariant Hilbert A-module $(\mathcal{E}, \gamma)$ there is an isomorphism $\Phi : \mathcal{E} \otimes_A (A \rtimes_\mu G) \xrightarrow{\cong} \mathcal{E} \rtimes_\mu G$, where A acts on $A \rtimes_\mu G$ via i_A^μ. It is given on elementary tensors $x \otimes f \in \mathcal{E} \odot C_c(G, A)$ by $\Phi(x \otimes f)(s) = x \cdot f(s)$. It is easily checked that Φ preserves the inner products and has dense image, hence extends to the desired isomorphism. If $\mathcal{E} = \mathcal{H} \otimes A$, we then get

$$(\mathcal{H} \otimes A) \rtimes_\mu G \cong (\mathcal{H} \otimes A) \otimes_A (A \rtimes_\mu G) \cong \mathcal{H} \otimes (A \rtimes_\mu G),$$

where the second isomorphism maps $\xi \otimes a \otimes f \in \mathcal{H} \odot A \odot C_c(G, A)$ to $x \otimes i_A^\mu(a) f$. Now it is not difficult to check on the generators that this isomorphism transfers the left action $(1_{\mathcal{H}} \otimes \mathrm{id}_A) \rtimes_\mu G$ of $A \rtimes_\mu G$ on $(\mathcal{H} \otimes A) \rtimes_\mu G$ to the action $(1_H \otimes i_A^\mu) \rtimes (\gamma \otimes i_G^\mu)$ on $\mathcal{H} \otimes (A \rtimes_\mu G)$. □

Proof of Theorem 5.6. It is enough to show that the quotient map $q_{\mathrm{r}}^\mu : A \rtimes_{\alpha,\mu} G \to A \rtimes_{\alpha,\mathrm{r}} G$ is a KK-equivalence. Then the result for the first map follows from composing the KK-equivalence in case $\rtimes_\mu = \rtimes_{\mathrm{u}}$ with the inverse of $[q_{\mathrm{r}}^\mu]$ in $KK(A \rtimes_{\alpha,\mathrm{r}} G, A \rtimes_{\alpha,\mu} G)$. We closely follow the arguments given in the proof of [25, Proposition 3.4].

So assume that $(\mathcal{H}, \gamma, T)$ is a cycle for $1_G \in KK^G(\mathbb{C}, \mathbb{C})$ such that γ is weakly contained in λ_G and let (A, α) be any G-C^*-algebra. Since $1_G \otimes_{\mathbb{C}} 1_A = 1_A$, it follows

that $1_A \in KK^G(A,A)$ is represented by the cycle $(\mathcal{H}\otimes A, 1_{\mathcal{H}}\otimes \mathrm{id}_A, \gamma\otimes\alpha, T\otimes 1)$. Using the isomorphisms

$$(\mathcal{H}\otimes A)\rtimes_\mu G \cong \mathcal{H}\otimes (A\rtimes_\mu G)$$

of Lemma 5.7, we see that $J_\mu(1_A) = 1_{A\rtimes_\mu G} \in KK(A\rtimes_\mu G, A\rtimes_\mu G)$ is represented by the cycle

$$\big(\mathcal{H}\otimes A\rtimes_\mu G, \Psi, T\otimes 1\big)$$

with $\Psi = (1_{\mathcal{H}}\otimes i_A^\mu)\rtimes(\gamma\otimes i_G^\mu)$. To see that this action factors through $A\rtimes_{\mathrm{r}} G$ we first observe that $C^*_{\mathrm{r}}(G)\otimes A\rtimes_\mu G$ acts on $\mathcal{H}\otimes A\rtimes_\mu G$ via $\gamma\otimes \mathrm{id}_{A\rtimes_\mu G}$, since γ is weakly contained in λ_G by assumption. Moreover, the covariant homomorphism

$$(1_{C^*_{\mathrm{r}}(G)}\otimes i_A^\mu)\rtimes(\lambda_G\otimes i_G^\mu) : A\rtimes_{\mathrm{u}} G \to \mathcal{M}(C^*_{\mathrm{r}}(G)\otimes A\rtimes_\mu G)$$

factors through $A\rtimes_{\mathrm{r}} G$, since this is already true if we replace $A\rtimes_\mu G$ by $A\rtimes_{\mathrm{u}} G$ (compare the discussion on KLQ-functors in §2.3). Now one checks on generators that

$$\Psi = (\gamma\otimes \mathrm{id}_{A\rtimes_\mu G})\circ\big((1_{C^*_{\mathrm{r}}(G)}\otimes i_A^\mu)\rtimes(\lambda_G\otimes i_G^\mu)\big),$$

hence the result. Thus, replacing the left action of $A\rtimes_\mu G$ by the left action of $A\rtimes_{\mathrm{r}} G$, we obtain an element $x_A\in KK(A\rtimes_{\mathrm{r}} G, A\rtimes_\mu G)$ such that $1_{A\rtimes_\mu G} = (q_{\mathrm{r}}^\mu)^*(x_A) = [q_{\mathrm{r}}^\mu]\otimes_{A\rtimes_{\mathrm{r}} G} x_A$. The converse equation $x_A\otimes_{A\rtimes_{\mathrm{u}} G}[q_{\mathrm{r}}^\mu] = 1_{A\rtimes_{\mathrm{r}} G}$ follows word for word as in [25, Proposition 3.4] and we omit further details. □

Remark 5.8. In case of a-T-menable groups a different line of argument could be used to obtain the above proposition. In fact, it is shown in [22, Theorems 8.5 and 8.6] that the elements $D\in KK^G(\mathcal{B},\mathbb{C})$ and $\eta\in KK^G(\mathbb{C},\mathcal{B})$ which implement the validity of the strong Baum-Connes conjecture can be chosen to be KK^G-equivalences. Since for proper actions the full and reduced crossed products (and hence all exotic crossed products) coincide, we can use the descents for $\rtimes_\mu$ and $\rtimes_{\mathrm{u}}$ to obtain the following chain of KK-equivalences for a given G-C^*-algebra (A,α):

$$A\rtimes_\mu G \sim_{KK^G} (A\otimes\mathcal{B})\rtimes_\mu G \cong (A\otimes\mathcal{B})\rtimes_{\mathrm{u}} G \sim_{KK^G} A\rtimes_{\mathrm{u}} G.$$

One can deduce from this that the quotient maps

$$A\rtimes_{\mathrm{u}} G \twoheadrightarrow A\rtimes_\mu G \twoheadrightarrow A\rtimes_{\mathrm{r}} G$$

are also KK-equivalences.

6. L^p examples

Some of the most interesting examples of exotic crossed products come from conditions on the decay of matrix coefficients. In this section, we use examples to explore some of the phenomena that can occur. There is no doubt much more to say here.

For a locally compact group G and $p\in[1,\infty]$, let $E_p\subseteq B(G)$ denote the weak*-closure of the intersection $L^p(G)\cap B(G)$. This is clearly an ideal in $B(G)$, and so gives rise to a KLQ-crossed product, and in particular a completion $C^*_{E_p}(G)$ of the group algebra. Recall from [6, Proposition 2.11 and Proposition 2.12][2] that $C^*_{E_p}(G)$ is always the reduced completion for $p\le 2$, and that $C^*_{E_p}(G) = C^*(G)$ for some $p<\infty$ if and only if G is amenable. Of course, since $E_\infty = B(G)$, we always get $C^*_{E_\infty}(G) = C^*(G)$.

[2]This reference only studies the discrete case, but the same proofs work for general locally compact groups.

Similarly, let $E_0 \subseteq B(G)$ denote the weak*-closure of $C_0(G) \cap B(G)$, which is also an ideal. For countable discrete groups, Brown and Guentner [6, Corollary 3.4] have shown that $E_0 = B(G)$ if and only if G is a-T-menable. In [24] Jolissaint extended this to general second countable[3] locally compact groups.

The following theorem records some properties of the algebras $C^*_{E_p}(G)$ for the free group on two generators.

Theorem 6.1. *Let $G = \mathbb{F}_2$ be the free group on two generators. Then one has the following results.*

(1) For $p \in [2, \infty]$ the ideals E_p, or equivalently the C^*-algebras $C^*_{E_p}(G)$, are all different (in the sense that the identity on $C_c(G)$ does not extend to isomorphisms of these algebras for different parameters p).

(2) The union of the ideals E_p for $p \in [2, \infty)$ is weak-dense in $B(G) = E_0 = E_\infty$.*

(3) The C^*-algebras $C^*_{E_p}(G)$, $p \in [2, \infty]$, all have the same K-theory (in the strong sense that the canonical quotient maps between them induce KK-equivalences).

Proof. Properties (1) and (2) follow from Okayasu's work [32]. Property (3) is a consequence of Theorem 5.6 since $\mathbb{F}_2$ is a-T-menable and the corresponding KLQ-functors are correspondence functors. □

It seems interesting to ask exactly which other groups have properties (1), (2) and (3) from Theorem 6.1. We now discuss this.

First, we look at property (1). If a group G has this property, then clearly G must be non-amenable, otherwise all the spaces E_p, $p \in \{0\} \cup [1, \infty]$, are the same. It is easy to see that property (1) for $\mathbb{F}_2$ implies that this property holds for all discrete groups that contain $\mathbb{F}_2$ as a subgroup. It is thus quite conceivable that it holds for all non-amenable discrete groups.

Property (1) also holds for some non-discrete groups: it was observed recently by Wiersma (see [37, §7]), using earlier results of Elias Stein, that (1) holds for $SL(2, \mathbb{R})$. Indeed, it is shown in [37, Theorem 7.3] that $p \mapsto E_p = \overline{B(G) \cap L^p(G)}^{w*}$ is a bijection between $[2, \infty]$ and the set of weak*-closed ideals $E \subseteq B(G)$ for $G = SL(2, \mathbb{R})$. However, property (1) does not hold for all non-amenable locally compact groups as the following example shows.

Example 6.2. Let G be a non-compact simple Lie group with finite center and rank at least two. It is well-known that such a group is non-amenable. Cowling [12, page 233][4] has shown that there exists $p \in (2, \infty)$ (depending on G) such that all non-trivial irreducible unitary representations of G have all their matrix coefficients in $L^p(G)$. Hence for all $q \in [p, \infty)$ we have that $C^*_{E_q}(G) = C^*_{E_p}(G)$. Indeed, this follows as the sets of irreducible representations of these C^*-algebras are the same: they identify canonically with the set of all non-trivial irreducible unitary representations of G.

[3]The techniques of Brown and Guentner, and of Jolissaint, in fact show that $E_0 = B(G)$ if and only if G has the *Haagerup approximation property* in the sense of [10, 1.1.1 (2)] without any second countability assumptions. The Haagerup approximation property is equivalent to all the other possible definitions of a-T-menability in the second countable case, but not in general: see [10, Section 1.1.1 and Theorem 2.1.1] for more details on this.

[4]See [21, Corollary 3.3.13] for more detail in the special case when $G = SL(n, \mathbb{R})$, $n \geq 3$.

We now look at property (2). As written, the property can only possibly hold for a-T-menable groups. Simple a-T-menable Lie groups of rank one (*i.e.* $SU(n,1)$ and $SO(n,1)$) have the property by already cited results of Cowling [12, page 233]. For discrete groups, one has the following sufficient condition for property (2) to hold.

Lemma 6.3. *Say G is a finitely generated discrete group, and fix a word length $l : G \to \mathbb{N}$. Assume moreover there exists a negative type function $\psi : G \to \mathbb{R}_+$ such that $\psi(g) \geq c \cdot l(g)$ for some $c > 0$ and all $g \in G$. Then the union $\cup E_p$ is weak*-dense in $B(G)$.*

Proof. Let ϕ be any element of $B(G)$. For $t > 0$ let $\phi_t(g) = e^{-t\psi(g)}$, which is positive type by Schoenberg's theorem. For any fixed $t > 0$ and all p suitably large, the function ϕ_t will be in $l^p(G)$: indeed this follows from the estimate on ψ and as there exists $b > 0$ such that $|\{g \in G \ : \ l(g) \leq r\}| \leq e^{br}$ for any $r > 0$ (the latter fact is an easy consequence of finite generation). Hence the functions $\phi \cdot \phi_t$ are all in some E_p (where p will in general vary as t does), and they converge pointwise, whence weakly, to ϕ. □

The condition in the lemma is closely related to the *equivariant compression* of G [20], and has been fairly well-studied. It is satisfied for example by finitely generated free groups, but also much more generally: for example, it can be easily deduced from [31] that any group acting properly cocompactly by isometries on a CAT(0) cube complex has this property, and from [5, Section 2.6] that any group acting properly cocompactly by isometries on a real hyperbolic space does. Note, however, that [2] implies that the lemma only applies to a subclass of finitely generated a-T-menable groups.

Finally, we look at property (3). By Theorem 5.6 it holds for all second countable K-amenable groups (in particular, all a-T-menable groups). The following example shows that it fails in general.

Example 6.4. Say $G = SP(n,1)$, and $n \geq 3$. Prudhon [35] has shown that there are infinitely many irreducible representations $(\pi, \mathcal{H}_\pi)$ of G – the so-called *isolated series* representations – such that there is a direct summand of $C^*(G)$ isomorphic to $\mathcal{K}(\mathcal{H}_\pi)$, and moreover that the corresponding direct summand $K_0(\mathcal{K}(\mathcal{H}_\pi)) \cong \mathbb{Z}$ of the group $K_0(C^*(G))$ is in the kernel of the natural map $\lambda_* : K_0(C^*(G)) \to K_0(C^*_{\mathrm{r}}(G))$. It follows from the previously cited work of Cowling [12, page 233] that any non-trivial irreducible representation of $SP(n,1)$ extends to $C^*_{E_p}(G)$ for some $p < \infty$. Combining all of this, it follows that there exists $p \in (2, \infty)$ such that $C^*_{E_p}(G)$ has a direct summand isomorphic to $\mathcal{K}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$, and that the corresponding direct summand $K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$ of $K_0(C^*_{E_p}(G))$ is in the kernel of the map on K-theory induced by the natural quotient $C^*_{E_p}(G) \to C^*_{E_2}(G)$. Hence property (3) fails for $SP(n,1)$ for $n \geq 3$.

One can deduce from work of Pierrot [34] that property (3) fails for $SL(4,\mathbb{C})$ and $SL(5,\mathbb{C})$ in a similar way.

The work of Prudhon and Pierrot cited above uses detailed knowledge of the representation theory of the groups involved. It would be interesting to have more directly accessible examples, or techniques that showed similar results for discrete groups. For example, it is natural to guess that similar failures of property (3) occur for discrete hyperbolic groups with property (T), and for lattices in higher rank simple Lie groups; proving such results seems to require new ideas, however.

7. Morita compatibility and the minimal exact correspondence functor

We now want to relate our results to the constructions of [4], in which a new version of the Baum-Connes conjecture is discussed. In fact, the authors consider a certain minimal exact and Morita compatible (in a sense explained below) crossed-product functor, which they call $\rtimes_{\mathcal{E}}$, and the new version of the Baum-Connes conjecture formulated by Baum, Guentner and Willett in that paper asserts that the Baum-Connes assembly map should always be an isomorphism if we replace reduced crossed products by $\rtimes_{\mathcal{E}}$ crossed products. We should note that if G is exact, then $\rtimes_{\mathcal{E}} = \rtimes_{\mathrm{r}}$, so that in this case the new conjecture coincides with the old one. However, all known counter examples for the original conjecture are due to the existence of non-exact groups and cannot be shown to be counterexamples to the reformulated conjecture. The most remarkable result of [4] shows that many such examples actually do satisfy the reformulated conjecture.

Note that Baum, Guentner and Willett also constructed a direct assembly map for the crossed-product functor $\rtimes_{\mathcal{E}}$ where they used the descent in E-theory instead of KK-theory, since at the time of writing it was not clear to them whether the KK-theory descent exists (*e.g.*, see the open question [4, Question 8.1 (vii)]). In this section we show that this is indeed the case, so that KK-methods do apply to the reformulation of the Baum-Connes conjecture.

We start the discussion by showing that the projection property is inherited by the infimum of a collection of crossed-product functors which satisfy this property. Recall from [4, Lemma 3.7] that, starting with a collection of crossed-product functors $\{\rtimes_\mu : \mu \in \Sigma\}$ (where we *do not* assume that Σ will be a set), a new crossed-product functor $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu_{\mathrm{inf}}} G$ can be constructed, which should be understood as the *infimum* of the functors in the collection $\{\rtimes_\mu : \mu \in \Sigma\}$. The crossed product $A \rtimes_{\alpha,\mu_{\mathrm{inf}}} G$ is the unique quotient of $A \rtimes_{\alpha,\mathrm{u}} G$ such that the set $(A \rtimes_{\alpha,\mu_{\mathrm{inf}}} G)\widehat{\ }$ of equivalence classes of irreducible representations (denoted $S_{\mu_{\mathrm{inf}}}(A)$ in [4]) is given by the formula

$$(A \rtimes_{\alpha,\mu_{\mathrm{inf}}} G)\widehat{\ } = \bigcap_{\mu\in\Sigma} (A \rtimes_{\alpha,\mu} G)\widehat{\ }, \tag{7.1}$$

where we view each dual space $(A \rtimes_{\alpha,\mu} G)\widehat{\ }$ as a subset of $(A \rtimes_{\alpha,\mathrm{u}} G)\widehat{\ }$ via composition with the quotient maps $q_{A,\mu} : A \rtimes_{\alpha,\mathrm{u}} G \to A \rtimes_{\alpha,\mu} G$. Alternatively, one may define $A \rtimes_{\alpha,\mu_{\mathrm{inf}}} G$ as the quotient $(A \rtimes_{\alpha,\mathrm{u}} G)/I_{\mu_{\mathrm{inf}}}$ with

$$I_{\mu_{\mathrm{inf}}} = \overline{\sum_{\mu\in\Sigma} I_\mu},$$

the closed sum of all ideals $I_\mu := \ker q_{A,\mu}$, $\mu \in \Sigma$. Note that this does not cause any set theoretic problems, since the collection of ideals $\{I_\mu : \mu \in \Sigma\}$ is a set.

It has been shown in [4, Lemma 3.7] that $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu_{\mathrm{inf}}} G$ is indeed a crossed-product functor. Moreover, it is shown in [4, Theorem 3.8] that taking the infimum of a collection of exact crossed-product functors will again give an exact crossed-product functor. We now show

Proposition 7.2. *Let $\{\rtimes_\mu : \mu \in \Sigma\}$ be a collection of correspondence crossed-product functors. Then its infimum $\rtimes_{\mu_{\mathrm{inf}}}$ will be a correspondence functor as well.*

We need a lemma:

Lemma 7.3. *Suppose that (A,α) is a G-algebra and $B \subseteq A$ is a G-invariant hereditary subalgebra. For each covariant representation (π, U) of (A, G, α) on some Hilbert space (or Hilbert module) $\mathcal{H}$ let us write $(\pi|_B, U)$ for the covariant representation of (B, G, α) which acts on the subspace (or submodule) $\mathcal{H}|_B := \pi(B)\mathcal{H}$ via the restriction $\pi|_B$ and the given unitary representation U. Then*

$$(B \rtimes_{\alpha,\mathrm{u}} G)\widehat{\ } = \{\pi|_B \rtimes U : \pi \rtimes U \in (A \rtimes_{\alpha,\mathrm{u}} G)\widehat{\ }, \pi(B) \neq \{0\}\}. \tag{7.4}$$

In particular, if $p \in \mathcal{M}(A)$ is a G-invariant projection, then

$$(pAp \rtimes_{\alpha,\mathrm{u}} G)\widehat{\ } = \{\pi|_{pAp} \rtimes U : \pi \rtimes U \in (A \rtimes_{\alpha,\mathrm{u}} G)\widehat{\ }, \pi(p) \neq \{0\}\}. \tag{7.5}$$

Proof. We first consider the ideal $I := \overline{ABA}$. It is then well known that

$$(I \rtimes_{\alpha,\mathrm{u}} G)\widehat{\ } = \{\pi|_I \rtimes U : \pi \rtimes U \in (A \rtimes_{\alpha,\mathrm{u}} G)\widehat{\ }, \pi(I) \neq \{0\}\}.$$

Note that we then have $\pi(I)\mathcal{H} = \mathcal{H}$ if $\pi(I) \neq \{0\}$, since $\pi \rtimes U$ is irreducible and $\pi(I)\mathcal{H}$ is $\pi \rtimes U$-invariant.

Note that $\pi(ABA) = \pi(A)\pi(B)\pi(A) = \{0\}$ if and only if $\pi(B) = \{0\}$, which follows by approximating $b \in B$ by $a_i b a_i$, where (a_i) runs through a bounded approximate unit of A. Using this and the above observation for the ideal $I = \overline{ABA}$, we may now assume that B is full in the sense that $\overline{ABA} = A$ and to show that in this situation we have

$$(B \rtimes_{\alpha,\mathrm{u}} G)\widehat{\ } = \{\pi|_B \rtimes U : \pi \rtimes U \in (A \rtimes_{\alpha,\mathrm{u}} G)\widehat{\ }\}.$$

For this we observe that (BA, α) is a G-equivariant $B - A$ equivalence bimodule with B- and A-valued inner products given by ${}_B\langle c \mid d\rangle = cd^*$ and $\langle c \mid d\rangle_A = c^*d$ for $c, d \in BA$. Thus, BA induces a bijection between the equivalence classes of irreducible covariant representations of (A, G, α) to those of (B, G, α) by sending a representation (π, U) to the BA-induced representation $(\mathrm{Ind}^{BA}\pi, \mathrm{Ind}^{BA} U)$ on the Hilbert space $(BA) \otimes_\pi \mathcal{H}$ on which $\mathrm{Ind}^{BA}\pi$ is given by the left action of B on the first factor BA and $\mathrm{Ind}^{BA} U = \alpha \otimes U$ (*e.g.*, see [11, 16]). Now observe that there is a unique unitary operator $V : BA \otimes_\pi \mathcal{H} \to \pi(B)\mathcal{H}$ given on elementary tensors by sending $ba \otimes \xi$ to $\pi(ba)\xi$, which clearly intertwines $(\mathrm{Ind}^{BA}\pi, \mathrm{Ind}^{BA} U)$ with $(\pi|_B, U)$. This proves (7.4) and (7.5) then follows from the fact that $\pi(pAp) \neq 0$ if and only if $\pi(p) \neq 0$. □

Corollary 7.6. *A given crossed-product functor $(A, \alpha) \to A \rtimes_{\alpha,\mu} G$ satisfies the projection property (hence is a correspondence functor) if and only if*

$$(pAp \rtimes_{\alpha,\mu} G)\widehat{\ } = \{\pi|_{pAp} \rtimes U : \pi \rtimes U \in (A \rtimes_{\alpha,\mu} G)\widehat{\ }, \pi(p) \neq \{0\}\}. \tag{7.7}$$

holds for every G-invariant projection $p \in \mathcal{M}(A)$.

Proof. Let $\tilde{p}$ denote the image of p in $\mathcal{M}(A \rtimes_\mu G)$ under the canonical map. Then $\tilde{p}(A \rtimes_\mu G)\tilde{p}$ is a corner of $A \rtimes_{\alpha,\mu} G$ which implies that

$$(\tilde{p}(A \rtimes_\mu G)\tilde{p})\widehat{\ } = \{(\pi \rtimes U)|_{\tilde{p}(A \rtimes_\mu G)\tilde{p}} : \pi \rtimes U \in (A \rtimes_{\alpha,\mu} G)\widehat{\ }, \pi \rtimes U(\tilde{p}) \neq 0\}.$$

A computation on the level of functions in $C_c(G, pAp)$ shows that $(\pi \rtimes U)|_{\tilde{p}(A \rtimes_\mu G)\tilde{p}}$ is the integrated form of $(\pi|_{pAp}, U)$. Since $\pi \rtimes U(\tilde{p}) = \pi(p)$ the homomorphism

$$\iota \rtimes G : pAp \rtimes_{\alpha,\mu} G \twoheadrightarrow \tilde{p}(A \rtimes_\mu G)\tilde{p} \subseteq A \rtimes_{\alpha,\mu} G$$

will be faithful if and only if equation (7.7) holds. □

Proof of Proposition 7.2. By Theorem 4.9 it suffices to show that if all functors $\rtimes_\mu$ satisfy the projection property, the same holds for $\rtimes_{\mu_{\inf}}$. So we may assume that equation (7.7) holds for each μ. But then it also holds for $\mu_{\inf}$, since for every G-invariant projection $p \in \mathcal{M}(A)$ we have

$$\begin{aligned}(pAp \rtimes_{\alpha,\mu_{\inf}} G)\widehat{\ } &= \bigcap_{\mu\in\Sigma} (pAp \rtimes_{\alpha,\mu} G)\widehat{\ } \\ &= \bigcap_{\mu\in\Sigma} \{\pi|_{pAp} \rtimes U : \pi \rtimes U \in (A \rtimes_{\alpha,\mu} G)\widehat{\ }, \pi(p) \neq \{0\}\} \\ &= \{\pi|_{pAp} \rtimes U : \pi \rtimes U \in (A \rtimes_{\alpha,\mu_{\inf}} G)\widehat{\ }, \pi(p) \neq \{0\}\}.\end{aligned}$$

Hence the result follows from Corollary 7.6. □

The following corollary gives a counterpart to the minimal exact Morita compatible crossed-product functor $(A,\alpha) \mapsto A \rtimes_{\alpha,\mathcal{E}} G$ as constructed in [4]. Since it is a correspondence functor, it allows a descent in KK-theory and the results of §5 will apply. We shall see below that, at least for second countable groups and on the category of separable G-algebras, the functor $\rtimes_{\mathcal{E}}$ coincides with the minimal exact correspondence functor $\rtimes_{\mathcal{E}_{\mathfrak{Corr}}}$ of

Corollary 7.8. *For every locally compact group G there exists a smallest exact correspondence crossed-product functor $(A,\alpha) \mapsto A \rtimes_{\alpha,\mathcal{E}_{\mathfrak{Corr}}} G$.*

Proof. We simply define $\rtimes_{\mathcal{E}_{\mathfrak{Corr}}}$ as the infimum of the collection of all exact correspondence functors $\rtimes_\mu$. By Proposition 7.2 it will be a correspondence functor and by [4, Theorem 3.8] it will be exact. □

The minimal exact correspondence functor $\rtimes_{\mathcal{E}_{\mathfrak{Corr}}}$ enjoys the following property:

Corollary 7.9. *Let (D,δ) be any unital G-C^*-algebra. Then for each G-C^*-algebra (A,α) the inclusion $j_A : A \to A \otimes_{\max} D$; $a \mapsto a \otimes 1$ descends to an injective *-homomorphism $j_A \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G : A \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G \to (A \otimes_{\max} D) \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$.*

In other words, we have $\rtimes_\mu = \rtimes_{\mu_D^{\max}}$ if $\mu = \mathcal{E}_{\mathfrak{Corr}}$. If D is exact, the same holds if we replace $\otimes_{\max}$ by $\otimes_{\min}$.

Proof. This follows from minimality of $\rtimes_{\mathcal{E}_{\mathfrak{Corr}}}$ and the (easily verified) fact that $\rtimes_{\mu_D^\nu}$ is exact if $\rtimes_\mu$ is exact and either: $\nu = \max$; or both $\nu = \min$ and D is exact. □

In order to see that the functor $\rtimes_{\mathcal{E}_{\mathfrak{Corr}}}$ coincides with the "minimal exact Morita compatible functor" $\rtimes_{\mathcal{E}}$, we need to recall the notion of Morita compatibility as defined in [4] and then compare it with the properties of our correspondence functors. If G is a locally compact group we write $\mathcal{K}_G := \mathcal{K}(\ell^2(\mathbb{N})) \otimes \mathcal{K}(L^2(G))$ equipped with the G-action $\mathrm{Ad}\Lambda$, where $\Lambda = 1 \otimes \lambda : G \to \mathcal{U}(\ell^2(\mathbb{N}) \otimes L^2(G))$ and λ denotes the left regular representation of G. Then there is a canonical isomorphism of maximal crossed products

$$\Phi : (A \otimes \mathcal{K}_G) \rtimes_{\alpha\otimes\mathrm{Ad}\Lambda,\mathrm{u}} G \xrightarrow{\cong} (A \rtimes_{\alpha,\mathrm{u}} G) \otimes \mathcal{K}_G$$

given by the integrated form of the covariant homomorphism $(i_A \otimes \mathrm{id}_{\mathcal{K}_G}, i_G \otimes \Lambda)$ of $(A \otimes \mathcal{K}_G, G, \alpha \otimes \mathrm{Ad}\Lambda)$ into $\mathcal{M}((A \rtimes_{\alpha,\mathrm{u}} G) \otimes \mathcal{K}_G)$. Then Baum, Guentner and Willett defined a crossed-product functor to be *Morita compatible* if the composition of Φ with the quotient map $(A \rtimes_{\alpha,\mathrm{u}} G) \otimes \mathcal{K}_G \to (A \rtimes_{\alpha,\mu} G) \otimes \mathcal{K}_G$ factors through an isomorphism

$$\Phi_\mu : (A \otimes \mathcal{K}_G) \rtimes_{\alpha\otimes\mathrm{Ad}\Lambda,\mu} G \xrightarrow{\cong} (A \rtimes_{\alpha,\mu} G) \otimes \mathcal{K}_G.$$

Proposition 7.10. *Suppose that G is a locally compact group and that $(A,\alpha)\mapsto A\rtimes_{\alpha,\mu}G$ is a crossed-product functor on the category of σ-unital C^*-algebras. Then the following three conditions are equivalent:*

(1) $\rtimes_\mu$ is Morita compatible as defined above.
(2) $\rtimes_\mu$ has the full projection property.
(3) $\rtimes_\mu$ is strongly Morita compatible.

Note that the assumption that all G-algebras are σ-unital is only used in the proof (1) $\Rightarrow$ (2). All other statements hold in full generality.

Proof. The proof of (2) $\Leftrightarrow$ (3) is exactly the same as the proof of (2) $\Leftrightarrow$ (4) in Lemma 4.12 and we omit it here.

(1) $\Rightarrow$ (2): Let (A,α) be a σ-unital G-algebra and let $p\in\mathcal{M}(A)$ be a G-invariant full projection. We need to show that the inclusion $\iota: pAp\hookrightarrow A$ descends to give a faithful inclusion $\iota\rtimes G: pAp\rtimes_\mu G\hookrightarrow A\rtimes_\mu G$. This will be the case if and only if

$$(\iota\rtimes G)\otimes \mathrm{id}_{\mathcal{K}_G}: pAp\rtimes_\mu G\otimes\mathcal{K}_G\to A\rtimes_\mu G\otimes\mathcal{K}_G$$

is faithful, and we shall now deduce from Morita compatibility that this is the case.

Let $\alpha\otimes\mathrm{Ad}\Lambda$ be the corresponding action of G on $A\otimes\mathcal{K}_G$. It follows from [30, Corollary 2.6] that there exists a G-invariant partial isometry $v\in\mathcal{M}(A\otimes\mathcal{K}_G)$ such that $v^*v=1_A\otimes 1_{\mathcal{K}_G}$ and $vv^*=p\otimes 1_{\mathcal{K}_G}$. This implies that the mapping

$$\mathrm{Ad}v^*: pAp\otimes\mathcal{K}_G\to A\otimes\mathcal{K}_G;\ (a\otimes k)\mapsto v^*(a\otimes k)v;$$

is a G-equivariant isomorphism. Hence it induces an isomorphism

$$\mathrm{Ad}v^*\rtimes G: (pAp\otimes\mathcal{K}_G)\rtimes_\mu G\xrightarrow{\cong}(A\otimes\mathcal{K}_G)\rtimes_\mu G.$$

Let $\tilde v$ denote the image of v in $\mathcal{M}((A\rtimes_{\alpha,\mu}G)\otimes\mathcal{K}_G)$ under the composition $\Phi_\mu\circ i_{A\otimes\mathcal{K}_G}$, where $i_{A\otimes\mathcal{K}_G}:\mathcal{M}(A\otimes\mathcal{K}_G)\to\mathcal{M}((A\otimes\mathcal{K}_G)\rtimes_\mu G))$ denotes the canonical map. Then $\tilde v$ is a partial isometry with $\tilde v^*\tilde v=1_{A\rtimes G\otimes\mathcal{K}}$ and $\tilde v\tilde v^*=\tilde p\otimes 1_{\mathcal{K}_G}$ where $\tilde p$ denotes the image of p in $\mathcal{M}(A\rtimes_\mu G)$. Consider the diagram

$$\begin{tikzcd}
(pAp\rtimes_{\alpha,\mu}G)\otimes\mathcal{K}_G \arrow[r, "\iota\rtimes G\otimes\mathrm{id}_{\mathcal{K}}"] & \tilde p(A\rtimes_{\alpha,\mu}G)\tilde p\otimes\mathcal{K}_G \arrow[r, "\mathrm{Ad}\tilde v^*"] & (A\rtimes_{\alpha,\mu}G)\otimes\mathcal{K}_G \\
(pAp\otimes\mathcal{K}_G)\rtimes_{\alpha\otimes\mathrm{Ad}\Lambda,\mu}G \arrow[u, "\Phi_\mu"] \arrow[rr, "\mathrm{Ad}v^*\rtimes G"] & & (A\otimes\mathcal{K}_G)\rtimes_{\alpha\otimes\mathrm{Ad}\Lambda,\mu}G \arrow[u, "\Phi_\mu"']
\end{tikzcd}$$

One easily checks on the generators that this diagram commutes. All maps but $\iota\rtimes G\otimes\mathrm{id}_{\mathcal{K}}$ are known to be isomorphisms, hence $\iota\rtimes G\otimes\mathrm{id}_{\mathcal{K}}$ must be an isomorphism as well.

We finally show (3) $\Rightarrow$ (1): We need to show that the covariant homomorphism $(i^\mu_A\otimes\mathrm{id}_{\mathcal{K}}, i^\mu_G\otimes\Lambda)$ of $(A\otimes\mathcal{K}_G, G, \alpha\otimes\mathrm{Ad}\Lambda)$ into $\mathcal{M}((A\rtimes_{\alpha,\mu}G)\otimes\mathcal{K}_G)$ integrates to give an isomorphism

$$\Phi_\mu:(A\otimes\mathcal{K}_G)\rtimes_{\alpha\otimes\mathrm{Ad}\Lambda,\mu}G\xrightarrow{\cong}(A\rtimes_{\alpha,\mu}G)\otimes\mathcal{K}_G.$$

Recall that we do have a corresponding isomorphism Φ_{u} for the universal crossed products. This will factor through the desired isomorphism if we can show that the map

$$\begin{aligned}\Phi^*_{\mathrm{u}}:\big((A\rtimes_{\alpha,\mathrm{u}}G)\otimes\mathcal{K}_G\big)^\frown &\to\big((A\otimes\mathcal{K}_G)\rtimes_{\alpha\otimes\mathrm{Ad}\Lambda,\mathrm{u}}G\big)^\frown\\ (\pi\rtimes U)\otimes\mathrm{id}_{\mathcal{K}_G} &\mapsto\big((\pi\rtimes U)\otimes\mathrm{id}_{\mathcal{K}_G}\big)\circ\Phi_{\mathrm{u}}\end{aligned}$$

induces a bijection between $\big((A \rtimes_{\alpha,\mu} G) \otimes \mathcal{K}_G\big)\widehat{\ }$ and $\big((A \otimes \mathcal{K}_G) \rtimes_{\alpha\otimes \mathrm{Ad}\Lambda,\mu} G\big)\widehat{\ }$. For this let (π, U) be any covariant representation of $A \rtimes_{\alpha,\mathrm{u}} G$ and let $(i_{A\otimes\mathcal{K}_G}, i_G)$ denote the canonical maps from $A \otimes \mathcal{K}_G$ and G into $\mathcal{M}\big((A \otimes \mathcal{K}_G) \rtimes_{\alpha\otimes \mathrm{Ad}\Lambda,\mu} G\big)$. Then one checks from the definition of Φ_{u} that

$$\big((\pi \rtimes U) \otimes \mathrm{id}_{\mathcal{K}}\big) \circ \Phi_{\mathrm{u}} \circ i_{A\otimes\mathcal{K}_G} = \pi \otimes \mathrm{id}_{\mathcal{K}_G} \quad \text{and} \quad \big((\pi \rtimes U) \otimes \mathrm{id}_{\mathcal{K}}\big) \circ \Phi_{\mathrm{u}} \circ i_G = U \otimes \Lambda.$$

So the result will follow if we can show that

$$\big((A \otimes \mathcal{K}_G) \rtimes_{\alpha\otimes \mathrm{Ad}\Lambda,\mu} G\big)\widehat{\ } = \{(\pi \otimes \mathrm{id}_{\mathcal{K}_G}) \rtimes (U \otimes \Lambda) : \pi \rtimes U \in (A \rtimes_{\alpha,\mu} G)\widehat{\ }\}. \tag{7.11}$$

To see this consider the G-equivariant $(A\otimes\mathcal{K}_G, \alpha\otimes\Lambda) - (A, \alpha)$ equivalence bimodule $(A\otimes L^2(\mathbb{N}\times G), \alpha\otimes\Lambda)$. By (3) it descends to give an $(A\otimes\mathcal{K}_G) \rtimes_{\alpha\otimes \mathrm{Ad}\Lambda,\mu} G - A \rtimes_{\alpha,\mu} G$ equivalence bimodule $(A\otimes L^2(\mathbb{N}\times G)) \rtimes_{\alpha\otimes\Lambda,\mu} G$. The corresponding bijection between the dual spaces given by induction via this module can be described on the level of covariant representations by inducing a covariant representation (π, U) of $A \rtimes_{\alpha,\mu} G$ via the G-equivariant module $A \otimes L^2(\mathbb{N} \times G)$ (see [11]). It acts on the Hilbert space $\big(A\otimes L^2(\mathbb{N}\times G)\big) \otimes_A \mathcal{H}_\pi$ via the covariant representation $\big(\mathrm{id}_{A\otimes\mathcal{K}_G} \otimes 1, (\alpha\otimes\Lambda)\otimes 1\big)$. But the isomorphism $\big(A \otimes L^2(\mathbb{N} \times G)\big) \otimes_A \mathcal{H}_\pi \xrightarrow{\cong} \mathcal{H}_\pi \otimes L^2(\mathbb{N} \times G)$ given on elementary tensors by $(a \otimes \xi) \otimes v \mapsto \pi(a)v \otimes \xi$ transforms $\big(\mathrm{id}_{A\otimes\mathcal{K}_G} \otimes 1, (\alpha \otimes \Lambda) \otimes 1\big)$ into $(\pi \otimes \mathrm{id}_{\mathcal{K}_G}, U \otimes \Lambda)$, which then gives (7.11) and completes the proof. □

Lemma 7.12. *Let G be a second countable group, and say $\rtimes_\mu$ is a crossed-product functor, defined on the category of* separable *G-C^*-algebras. For each G-C^*-algebra A, let $(A_i)_{i\in I}$ be the net of G-invariant separable sub C^*-algebras of A ordered by inclusion. Define $A \rtimes_{\mathrm{sep},\mu} G$ by*

$$A \rtimes_{\mathrm{sep},\mu} G := \lim_{i\in I} A_i \rtimes_\mu G.$$

Then $A \rtimes_{\mathrm{sep},\mu} G$ is a completion of $C_c(G, A)$ with the following properties.

1. *$(A, \alpha) \mapsto A \rtimes_{\alpha,\mathrm{sep},\mu} G$ is a crossed-product functor for G.*
2. *If $\rtimes_\mu$ has the ideal property, the same holds for $\rtimes_{\mathrm{sep},\mu}$.*
3. *If $\rtimes_\mu$ is a correspondence functor, the same holds for $\rtimes_{\mathrm{sep},\mu}$.*
4. *If $\rtimes_\mu$ is exact, the same holds for $\rtimes_{\mathrm{sep},\mu}$.*
5. *If $\rtimes_\mu$ is Morita compatible, then the same holds for $\rtimes_{\mathrm{sep},\mu}$.*
6. *If $\rtimes_\nu$ is any crossed-product functor extending $\rtimes_\mu$ to all G-C^*-algebras, then $\rtimes_{\mathrm{sep},\mu} \geq \rtimes_\nu$.*

Proof. Note first that as G is second countable any $f \in C_c(G, A)$ has image in some second countable G-invariant subalgebra of A, and thus $A \rtimes_{\mathrm{sep},\mu} G$ contains $C_c(G, A)$ as a dense subset as claimed. It is automatically smaller than the maximal completion, and it is larger than the reduced as the canonical maps

$$A_i \rtimes_\mu G \to A_i \rtimes_{\mathrm{r}} G \to A \rtimes_{\mathrm{r}} G$$

give a compatible system, whence there is a map $A \rtimes_{\mathrm{sep},\mu} G \to A \rtimes_{\mathrm{r}} G$ (which is equal to the identity on $C_c(G, A)$) by the universal property of the direct limit.

(1) Functoriality follows from functoriality of the direct limit construction.

(2) If I is an ideal in A, then for each separable G-invariant C^*-subalgebra A_i of A as in the definition of $\rtimes_{\mathrm{sep},\mu}$, define $I_i = I \cap A_i$. Then the directed system $(I_i)_{i\in I}$ is cofinal in the one used to define $I \rtimes_{\mathrm{sep},\mu} G$, whence

$$I \rtimes_{\mathrm{sep},\mu} G = \lim_i (I_i \rtimes_\mu G).$$

The result now follows as each map $I_i \rtimes_\mu G \to A_i \rtimes_\mu G$ is injective by the ideal property for $\rtimes_\mu$, and injectivity passes to direct limits.

(3) We prove the projection property. Let p be a G-invariant projection in the multiplier algebra of A. The argument is similar to that of part (2). The only additional observation needed is that if we set B_i to be the C^*-subalgebra of A generated by $A_i p$ and A_i, then the net $(B_i)_{i\in I}$ is cofinal in the net defining $A \rtimes_{\mathrm{sep},\mu} G$. Moreover, p is in the multiplier algebra of each B_i, whence each of the injections $pB_ip \to B_i$ induces an injection on $\rtimes_\mu$ crossed products by the projection property for $\rtimes_\mu$. The result follows now on passing to the direct limit of the inclusions

$$pB_ip \rtimes_\mu G \to B_i \rtimes_\mu G.$$

(4) This is similar to parts (2) and (3). Given a short exact sequence

$$0 \to I \to A \to B \to 0$$

with quotient map $\pi : A \to B$, let (A_i) be the net of separable G-invariant subalgebras of A, and consider the net of short exact sequences

$$0 \to A_i \cap I \to A_i \to \pi(A_i) \to 0.$$

The nets $(A_i \cap I)$ and $(\pi(A_i))$ are cofinal in the nets defining the $\rtimes_{\mathrm{sep},\mu}$ crossed products for I and B respectively. On the other hand, exactness of μ gives that all the sequences

$$0 \to (A_i \cap I) \rtimes_\mu G \to A_i \rtimes_\mu G \to \pi(A_i) \rtimes_\mu G \to 0$$

are exact. The result follows as exactness passes to direct limits.

(5) This again follows a similar pattern: the only point to observe is that the collection of separable G-invariant C^*-algebras of $A \otimes \mathcal{K}_G$ of the form $A_i \otimes \mathcal{K}_G$, where A_i is a separable G-invariant C^*-subalgebra of A, is cofinal in the direct limit defining $(A \otimes \mathcal{K}_G) \rtimes_{\mathrm{sep},\mu} G$.

(6) From functoriality of $\rtimes_\nu$, there is a compatible system of *-homomorphisms

$$A_i \rtimes_\mu G = A_i \rtimes_\nu G \to A \rtimes_\nu G.$$

From the universal property of the direct limit there is a *-homomorphism $A \rtimes_{\mathrm{sep},\mu} G \to A \rtimes_\nu G$ which is clearly the identity on $C_c(G, A)$, completing the proof. □

Example 7.13. Every non-amenable second countable group has a crossed product $\rtimes_\mu$ for which the crossed-product functor $\rtimes_{\mathrm{sep},\mu}$ (in which we restrict $\rtimes_\mu$ to the category of separable G-algebras) does not coincide with $\rtimes_\mu$. For this, we use the construction of §2.5. Let $\mathcal{H}$ be a Hilbert space with uncountable Hilbert space dimension. Let $\mathcal{S}$ denote the set of all separable subalgebras of $\mathcal{K}(\mathcal{H})$, all equipped with the trivial G-action. We define a crossed-product functor $\rtimes_\mu$ by defining $A \rtimes_\mu G$ as the completion of $C_c(G, A)$ by the norm

$$\|f\|_\mu := \max\big\{\|f\|_{\mathrm{r}}, \sup\{\|\phi \circ f\|_{B\rtimes_{\mathrm{u}}G} : B \in \mathcal{S}, \phi \in \mathrm{Hom}^G(A, B)\}\big\},$$

where $\mathrm{Hom}^G(A, B)$ denotes the set of G-equivariant *-homomorphisms from A to B. As explained in §2.5, this is a functor.

Now consider $A = \mathcal{K}(\mathcal{H})$ with the trivial G-action. Then $\mathcal{K}(\mathcal{H}) \rtimes_\mu G = \mathcal{K}(\mathcal{H}) \rtimes_{\mathrm{r}} G \cong \mathcal{K}(\mathcal{H}) \otimes C^*_{\mathrm{r}}(G)$, since there are no nontrivial homomorphisms from $\mathcal{K}(\mathcal{H})$ into any of its separable subalgebras. On the other hand we have $B_i \rtimes_\mu G \cong B \rtimes_{\mathrm{u}} G$ for

every separable subalgebra $B \subseteq \mathcal{K}(\mathcal{H})$, which implies $\mathcal{K}(\mathcal{H}) \rtimes_{\mathrm{sep},\mu} G \cong \mathcal{K}(\mathcal{H}) \rtimes_{\mathrm{u}} G \cong \mathcal{K}(\mathcal{H}) \otimes C^*(G)$.

Corollary 7.14. *The following functors from the category of separable G-C^*-algebras to the category of C^*-algebras are the same.*

1. *The restriction to separable G-algebras of the minimum $\rtimes_{\mathcal{E}_{\mathfrak{Corr}}}$ over all exact correspondence functors defined for all G-C^*-algebras.*
2. *The restriction to separable G-algebras of the minimum $\rtimes_{\mathcal{E}}$ over all exact Morita compatible functors defined for all G-C^*-algebras.*
3. *The minimum over all exact correspondence functors defined for separable G-C^*-algebras.*
4. *The minimum over all exact Morita compatible functors defined for separable G-C^*-algebras.*

Proof. Label the crossed products (defined on separable G-C^*-algebras) appearing in the points above as $\rtimes_1$, $\rtimes_2$, $\rtimes_3$ and $\rtimes_4$. By (2) $\Rightarrow$ (1) of Proposition 7.10 (which holds in full generality) we clearly have

$$\rtimes_1 \geq \rtimes_2 \geq \rtimes_4, \text{ and } \rtimes_1 \geq \rtimes_3 \geq \rtimes_4.$$

It thus suffices to show that $\rtimes_4 \geq \rtimes_1$. Indeed, let $\rtimes_{\mathrm{sep},4}$ be the extension given by Lemma 7.12. Then $\rtimes_{\mathrm{sep},4}$ is an exact and Morita compatible functor on the category of σ-unital G-C^*-algebras, whence it is an exact correspondence functor on this category by Proposition 7.10. Hence in particular $\rtimes_4$ was actually an exact correspondence functor on the category of separable G-algebras to begin with, and so $\rtimes_{\mathrm{sep},4}$ is an exact correspondence functor on the category of all G-algebras. It is thus one of the functors that $\rtimes_1$ is the minimum over, so $\rtimes_1 \leq \rtimes_{\mathrm{sep},4}$ on the category of all G-algebras, and in particular $\rtimes_1 \leq \rtimes_4$ on the category of separable G-algebras as required. □

Remark 7.15. The above corollary shows that (at least for second countable groups and separable G-algebras) the reformulation of the Baum-Connes conjecture in [4] is equivalent to the statement that

$$\mathrm{as}^{\mathcal{E}_{\mathfrak{Corr}}}_{(A,G)} : K^{\mathrm{top}}_*(G;A) \mapsto K_*(A \rtimes_{\alpha,\mathcal{E}_{\mathfrak{Corr}}} G)$$

is always an isomorphism, where $\rtimes_{\mathcal{E}_{\mathfrak{Corr}}}$ is the minimal exact correspondence functor. By the results of §5 we can use the full force of equivariant KK-theory to study this version of the conjecture.

8. Remarks and questions

8.1. **Crossed products associated to $UC_b(G)$.** We only know one general construction of exact correspondence functors. Fix a unital G-algebra (D,δ). Let (A,α) be a G-algebra. The *-homomorphism

$$A \to A \otimes_{\max} D, \quad a \mapsto a \otimes 1$$

induces an injective *-homomorphism

$$C_c(G,A) \to (A \otimes_{\max} D) \rtimes_{\alpha\otimes\delta,\mathrm{u}} G.$$

Recall from Section 2.4 that the crossed product $A \rtimes_{\mathrm{u}^{\max}_D} G$ is by definition the closure of the image of $C_c(G,A)$ inside $(A \otimes D) \rtimes_{\alpha\otimes\delta,\mathrm{u}} G$.

The crossed product $\rtimes_{\mathrm{u}_D^{\max}}$ is always exact by [4, Lemma 5.4], and always a correspondence functor by Corollary 4.20. Here we will show that the collection

$$\{\rtimes_{\mathrm{u}_D^{\max}} \ : \ (D,\delta) \text{ a } G\text{-algebra}\}$$

of exact correspondence functors – which are the *only* exact correspondence functors we know for general G – has a concrete minimal element.

Lemma 8.1. *Let $UC_b(G)$ denote the C^*-algebra of bounded, left-uniformly continuous complex-valued functions on G, equipped with the G-action induced by translation.*

Then for any unital G-algebra D, $\rtimes_{\mathrm{u}_D^{\max}} \geq \rtimes_{\mathrm{u}_{UC_b(G)}^{\max}}$.

Proof. Consider any state $\phi : D \to \mathbb{C}$, and note that for any $d \in D$, the function $d_\phi : G \to \mathbb{C}$ defined by $d_\phi : g \mapsto \phi(\delta_g(d))$ is bounded and uniformly continuous, hence an element of $UC_b(G)$. Consider the map

$$\Phi : D \to UC_b(G), \quad d \mapsto d_\phi.$$

This is clearly equivariant, unital and positive. Moreover, positive maps of C^*-algebras with commutative codomains are automatically completely positive: indeed, post-composing with multiplicative linear functionals reduces this to showing that a state is a completely positive map to $\mathbb{C}$, and this follows from the GNS-construction which shows in particular that a state is a compression of a *-homomorphism by a one-dimensional projection. Hence Φ is completely positive. Now consider the diagram

$$\begin{array}{ccc} (A \otimes_{\max} D) \rtimes_{\mathrm{u}} G & \xrightarrow{(1\otimes\Phi)\rtimes_{\mathrm{u}} G} & (A \otimes_{\max} UC_b(G)) \rtimes_{\mathrm{u}} G \ , \\ \uparrow{\scriptstyle \iota_D} & & \uparrow{\scriptstyle \iota_{UC_b(G)}} \\ A \rtimes_{\mathrm{u}_D^{\max}} G & & A \rtimes_{\mathrm{u}_{UC_b(G)}^{\max}} G \end{array}$$

where the vertical maps are the injections given by definition of the crossed products on the bottom row, and the top line exists by Theorem 4.9 and the fact that $1 \otimes \Phi$ is completely positive. As Φ is unital, the composition

$$(1 \otimes \Phi) \rtimes_{\mathrm{u}} G \circ \iota_D : A \rtimes_{\mathrm{u}_D^{\max}} G \to \iota_{UC_b(G)}(A \rtimes_{\mathrm{u}_{UC_b(G)}^{\max}} G)$$

identifies with a *-homomorphism from $A \rtimes_{\mathrm{u}_D^{\max}} G$ to $A \rtimes_{\mathrm{u}_{UC_b(G)}^{\max}} G$ that extends the identity map on $C_c(G, A)$. This gives the desired conclusion. □

The crossed product $\rtimes_{\mathrm{u}_{UC_b(G)}^{\max}}$ has another interesting property. Indeed, recall the following from [1, Section 3].

Definition 8.2. A locally compact group G is *amenable at infinity* if it admits an amenable action on a compact topological space.

Lemma 8.3. *Say G is amenable at infinity. Then for any G-algebra (A, α),*

$$A \rtimes_{\alpha,\mathrm{u}_{UC_b(G)}^{\max}} G = A \rtimes_{\alpha,\mathrm{r}} G.$$

Proof. If G is amenable at infinity, then [1, Proposition 3.4] implies that the action of G on the spectrum X of $C_{ub}(G)$ is amenable. Hence for any G-algebra A, the

tensor product $A \otimes_{\max} UC_b(G)$ is a G-$C(X)$ algebra for an amenable G-space X. The result follows from [1, Theorem 5.4], which implies that

$$(A \otimes_{\max} UC_b(G)) \rtimes_{\mathrm{u}} G = (A \otimes_{\max} UC_b(G)) \rtimes_{\mathrm{r}} G. \qquad \square$$

A group that is amenable at infinity is always exact [1, Theorem 7.2]. For discrete groups, the converse is true by [33], but this is an open question in general; nonetheless, many exact groups, for example all almost connected groups [1, Proposition 3.3], are known to be amenable at infinity.

To summarise, we have shown that $\rtimes_{\mathrm{u}^{\max}_{UC_b(G)}}$ is an exact correspondence functor; that it is minimal among a large family of exact correspondence functors; and that for many (and possibly all) exact groups, it is equal to the reduced crossed product. The following question is thus very natural.

Question 8.4. Is the crossed product $\rtimes_{\mathrm{u}^{\max}_{UC_b(G)}}$ equal to the minimal exact correspondence crossed product $\rtimes_{\mathcal{E}_{\mathfrak{corr}}}$?

There are other natural questions one could ask about $\rtimes_{\mathrm{u}^{\max}_{UC_b(G)}}$: for example, is it a KLQ functor 'in disguise'? One could also ask this about any of the functors $\rtimes_{\mathrm{u}^{\max}_{D}}$.

8.2. **Questions about L^p functors.** Recall from Section 6 that E_p denotes the weak*-closure of $B(G) \cap L^p(G)$ in $B(G)$ for $p \in [2, \infty]$, and E_0 is the weak*-closure of $B(G) \cap C_0(G)$. The following questions about these ideals and the corresponding group algebras and crossed products seem natural and interesting.

(1) Say G is an exact group. Are all of the functors $\rtimes_{E_p}$ exact? More generally, if G is an exact group, are all KLQ-crossed products exact? Note that Example 3.5 shows that any non-amenable exact group admits a non-exact crossed-product functor.
(2) Are there non-exact groups and $p \in \{0\} \cup (2, \infty)$ such that $\rtimes_{E_p}$ is exact?
(3) For which locally compact groups does the canonical quotient $q : C^*_{E_0}(G) \to C^*_{\mathrm{r}}(G)$ induce an isomorphism on K-theory (one could also ask this for other p)? This cannot be true in general by Example 6.4, but we do not know any discrete groups for which it fails.
(4) For which groups G are all the exotic group C^*-algebras $C^*_{E_p}(G)$ different? Example 6.2 shows that this cannot be true for general non-amenable groups, but it could in principal be true for all non-amenable discrete groups.
(5) For which groups is E_0 the weak*-closure of $\cup_{p<\infty} E_p$? This holds for all simple Lie groups with finite center by results of Cowling [12], and we showed it holds for a fairly large class of discrete groups in Lemma 6.3. Conceivably, it could hold in full generality. More generally, one can ask: is E_q is the weak*- closure of $\cup_{p<q} E_p$ for all $q \in [2, \infty)$? Similarly, is it true that $E_p = \cap_{q>p} E_q$ for all $p \in [2, \infty)$? Okayasu [32, Corollary 3.10] has shown that this is true for $\mathbb{F}_2$ and Cowling, Haagerup and Howe [13] have shown that $E_2 = \cap_{p>2} E_p$ is always true. Not much else seems to be known here, however.

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