

A non-amenable groupoid whose maximal and reduced C^* -algebras are the same

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Abstract

We construct a locally compact groupoid with the properties in the title. Our example is based closely on constructions used by Higson, Lafforgue, and Skandalis in their work on counterexamples to the Baum-Connes conjecture. It is a bundle of countable groups over the one point compactification of the natural numbers, and is Hausdorff, second countable and étale with compact unit space.

1 Introduction

Say G is a locally compact group, and $C_c(G)$ is the convolution $*$ -algebra of continuous compactly supported complex-valued functions on G . In general $C_c(G)$ has many interesting C^* -algebra completions, but the two most important are: the maximal completion $C_{\max}^*(G)$, which is the completion taken over (the integrated forms of) all unitary representations of G ; and the reduced completion $C_{\text{red}}^*(G)$, which is the completion of $C_c(G)$ for the left regular representation on $L^2(G)$. An important theorem of Hulanicki [5] says that $C_{\max}^*(G) = C_{\text{red}}^*(G)$ if and only if G is amenable.

Now, say G is a locally compact, Hausdorff groupoid. To avoid measure-theoretic complications in the statements below, we will assume that G is étale. Much as in the case of groups, there is a canonically associated convolution $*$ -algebra $C_c(G)$ of continuous compactly supported functions on G , and this $*$ -algebra completes in a natural way to maximal and reduced C^* -algebras $C_{\max}^*(G)$ and $C_{\text{red}}^*(G)$. There is also a natural notion of topological amenability. See [10] and [1] for comprehensive treatments of groupoids and their C^* -algebras, and [11, Section 2.3] and [3, Section 5.6] for self-contained introductions covering only the (much simpler) étale case.

For groupoids, one has the following analogue of one direction of Hulanicki's theorem, for which we refer to work of Anantharaman-Delaroche and Renault [1, Theorem 6.1.8]¹.

¹Anantharaman-Delaroche and Renault actually prove this result in the much more sophisticated case that G is not étale, and thus need additional assumptions about Haar systems. See [3, Corollary 5.6.17] for a proof of Theorem 1.1 as stated here.

Theorem 1.1. *Say G is a locally compact, Hausdorff, étale groupoid. If G is topologically amenable, then $C_{\max}^*(G) = C_{\text{red}}^*(G)$.*

In this note, we show that the converse to this result is false, so Hulanicki's theorem does not extend to groupoids.

Theorem 1.2. *There exist locally compact, Hausdorff, second countable, étale groupoids with compact unit space that are not topologically amenable but that satisfy $C_{\max}^*(G) = C_{\text{red}}^*(G)$.*

The examples we give are a slight adaptation of counterexamples to the Baum-Connes conjecture for groupoids constructed by Higson, Lafforgue, and Skandalis [4, Section 2]. The essential extra ingredient needed is *property FD* of Lubotzky and Shalom [8]. Our examples are of a particularly simple form, and are in fact a bundle of groups: see Section 2 below for details of all this.

The existence of examples as in Theorem 1.2 is a fairly well-known open question (see for example [1, Remark 6.1.9], [11, Section 4.2], or [3, page 162]) and the answer did not seem to be known to experts. We thus thought Theorem 1.2 was worth publicizing despite the similarity to existing constructions. We should remark, however, that our results seem to have no bearing on the existence of transformation groupoids, or of principal groupoids, with the properties in Theorem 1.2. In particular, we cannot say anything about whether or not equality of maximal and reduced crossed products for a group action implies (topological) amenability of the action. See Section 3 for some comments along these lines.

This paper is structured as follows: Section 2 recalls the construction of Higson, Lafforgue and Skandalis, and proves Theorem 1.2; Section 3 collects some comments and questions about exactness, transformation groupoids, and coarse groupoids.

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2 Main result

We first recall a construction of a class of groupoids from [4, Section 2]; some groupoids from this class will have the properties in Theorem 1.2. The starting point for this construction is the following data.

Definition 2.1. Let Γ be a discrete group. An *approximating sequence* for Γ is a sequence (K_n) of subgroups of Γ with the following properties.

- (i) Each K_n is a normal, finite index subgroup of Γ .

- (ii) The sequence is nested: $K_n \supseteq K_{n+1}$ for all n .
- (iii) The intersection of the sequence is trivial: $\bigcap_n K_n = \{e\}$.

An *approximated group* is a pair $(\Gamma, (K_n))$ consisting of a discrete group together with a fixed approximating sequence.

Here then is the construction of Higson, Lafforgue, and Skandalis that we will use.

Definition 2.2. Let $(\Gamma, (K_n))$ be an approximated group, and for each n , let $\Gamma_n := \Gamma/K_n$ be the associated quotient group, and $\pi_n : \Gamma \rightarrow \Gamma_n$ the quotient map. For convenience, we also write $\Gamma = \Gamma_\infty$ and $\pi_\infty : \Gamma \rightarrow \Gamma$ for the identity map. As a set, define

$$G := \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} \{n\} \times \Gamma_n.$$

Put the topology on G that is generated by the following open sets.

- (i) For each $n \in \mathbb{N}$ and $g \in \Gamma_n$ the singleton $\{(n, g)\}$.
- (ii) For each fixed $g \in \Gamma$ and $N \in \mathbb{N}$, the set $\{(n, \pi_n(g)) \mid n \in \mathbb{N} \cup \{\infty\}, n > N\}$.

Finally equip G with the groupoid operations coming from the group structure on the ‘fibres’ over each n : precisely, the unit space is identified with the subspace $\{(n, g) \in G \mid g = e\}$; the source and range maps are defined by $r(n, g) = s(n, g) = (n, e)$; and composition and inverse are defined by $(n, g)(n, h) := (h, gh)$ and $(n, g)^{-1} := (n, g^{-1})$.

It is not difficult to check that G as constructed above is a locally compact, Hausdorff, second countable, étale groupoid. Moreover the unit space $G^{(0)}$ of G naturally identifies with the one-point compactification of \mathbb{N} . We call G the *HLS groupoid* associated to the approximated group $(\Gamma, (K_n))$.

In the rest of this section, we will characterise precisely when HLS groupoids as above are amenable, and when their maximal and reduced groupoid C^* -algebras coincide. We will then use results of Lubotzky and Shalom to show that examples as in Theorem 1.2 that are ‘between’ these two properties exist.

Our first lemma characterises when an HLS groupoid G is amenable. We first recall definitions of amenability that are appropriate for our purposes: for the groupoid definition, compare [3, Definition 5.6.13] and [1, Proposition 2.2.13 (ii)]; for the group definition, see for example [3, Definition 2.6.3 and Theorem 2.6.8].

Definition 2.3. Let G be a locally compact, Hausdorff, étale groupoid. G is *amenable* if for any compact subset K of G and $\epsilon > 0$ there exists a continuous, compactly supported function $\eta : G \rightarrow [0, 1]$ such that for all $g \in K$

$$\left| \sum_{h \in G: s(h)=r(g)} \eta(h) - 1 \right| < \epsilon \quad \text{and} \quad \sum_{h \in G: s(h)=r(g)} |\eta(h) - \eta(hg)| < \epsilon.$$

Let Γ be a discrete group. Γ is *amenable* if for any finite subset F of G and $\delta > 0$ there exists a finitely supported function $\xi : \Gamma \rightarrow [0, 1]$ such that for all $g \in F$

$$\sum_{g \in \Gamma} |\xi(hg) - \xi(h)| < \delta.$$

Lemma 2.4. *Let G be the HLS groupoid associated to an approximated group $(\Gamma, (K_n))$. Then G is (topologically) amenable if and only if Γ is amenable.*

Proof. This is immediate from [1, Examples 5.1.3 (1)]. For the convenience of the reader, however, we also provide a direct proof.

Assume G is amenable, and let a finite subset F of Γ and $\delta > 0$ be given. Let K be the finite (hence compact) subset $\{\infty\} \times F$ of G , and let $\eta : G \rightarrow [0, 1]$ be as in the definition of amenability for G for the compact subset K and error tolerance $\epsilon < \delta/(1 + \delta)$. Write $M = \sum_{g \in \Gamma} \eta(\infty, g)$ (and note that this is at least $1 - \epsilon$) and define $\xi : \Gamma \rightarrow [0, 1]$ by

$$\xi(g) = \frac{1}{M} \eta(\infty, g);$$

it is not difficult to check this works.

Conversely, assume that Γ is amenable, and let a compact subset K of G and $\epsilon > 0$ be given. Let F be a finite subset of Γ such that $\pi_n(F) \supseteq K \cap \{n\} \times \Gamma_n$ for all n (compactness of K implies that such a set exists), and let $\xi : \Gamma \rightarrow [0, 1]$ be as in the definition of amenability for this F and error tolerance $\delta = \epsilon$. Define $\eta : G \rightarrow [0, 1]$ by

$$\eta(n, g) = \sum_{h \in \pi_n^{-1}(g)} \xi(h);$$

it is again not difficult to check that this works. □

Our next goal is to characterise when the maximal and reduced groupoid C^* -algebras of an HLS groupoid G are equal. We will not recall the general definitions of the maximal and reduced C^* -algebras for an étale groupoid here - for this we refer the reader to [11, Definition 2.3.18] or [3, pages 159-160] - but we state explicitly what they come down to in our situation.

Definition 2.5. Let G be an HLS groupoid associated to the approximated group $(\Gamma, (K_n))$. Let $C_c(G)$ denote the space of continuous, compactly supported, complex-valued functions on G equipped with the convolution product and involution defined by

$$(f_1 f_2)(n, g) := \sum_{h \in \Gamma_n} f_1(n, gh^{-1}) f_2(n, h), \quad f^*(n, g) := \overline{f(n, g^{-1})}.$$

The *maximal C^* -algebra of G* , denoted $C_{\max}^*(G)$, is the completion of $C_c(G)$ for the norm

$$\|f\|_{\max} := \sup\{\|\rho(f)\| \mid \rho : C_c(G) \rightarrow \mathcal{B}(H) \text{ a } *- \text{homomorphism}\}.$$

For $n \in \mathbb{N} \cup \{\infty\}$, define a $*$ -representation $\rho_n : C_c(G) \rightarrow \mathcal{B}(l^2(\Gamma_n))$ by the formula

$$(\rho_n(f)\xi)(g) = \sum_{h \in \Gamma_n} f(n, gh^{-1})\xi(h).$$

The *reduced C^* -algebra of G* , denoted $C_{\text{red}}^*(G)$, is the completion of $C_c(G)$ for the norm

$$\|f\|_{\text{red}} := \sup\{\|\rho_n(f)\| \mid n \in \mathbb{N} \cup \{\infty\}\}.$$

Lemma 2.6. *Let G be the HLS groupoid associated to an approximated group $(\Gamma, (K_n))$. For each $n \in \mathbb{N} \cup \{\infty\}$, let*

$$\lambda_n : \mathbb{C}[\Gamma] \rightarrow \mathcal{B}(l^2(\Gamma_n))$$

denote the quasi-regular representation induced by the left multiplication action of Γ on Γ_n .

Then we have $C_{\text{max}}^(G) = C_{\text{red}}^*(G)$ if and only if the maximal norm on $\mathbb{C}[\Gamma]$ equals the norm defined by*

$$\|x\| := \sup_{n \in \mathbb{N} \cup \{\infty\}} \|\lambda_n(x)\| \quad (1)$$

Proof. Let $G_{\mathbb{N}}$ denote the open subgroupoid of G consisting of all pairs (n, g) with $n \in \mathbb{N}$, and let $C_c(G_{\mathbb{N}})$ denote the $*$ -subalgebra of $C_c(G)$ consisting of functions that are supported on a subset of $G_{\mathbb{N}}$. The quotient $*$ -homomorphism $\pi : C_c(G) \rightarrow \mathbb{C}[\Gamma]$ defined by restriction to the fibre at infinity has kernel $C_c(G_{\mathbb{N}})$, and thus there is a short exact sequence of $*$ -algebras

$$0 \longrightarrow C_c(G_{\mathbb{N}}) \longrightarrow C_c(G) \xrightarrow{\pi} \mathbb{C}[\Gamma] \longrightarrow 0. \quad (2)$$

Note that as a $*$ -algebra $C_c(G_{\mathbb{N}})$ is isomorphic to the $*$ -algebra direct sum $\bigoplus_{n \in \mathbb{N}} \mathbb{C}[\Gamma_n]$, and thus to an infinite (algebraic) direct sum of matrix algebras. It follows that $C_c(G_{\mathbb{N}})$ has a unique C^* -algebra completion, which identifies canonically with the C^* -algebra direct sum $\bigoplus_{n \in \mathbb{N}} C_{\text{red}}^*(\Gamma_n)$. On the other hand, if π is as in line (2), then the $C_{\text{max}}^*(G)$ and $C_{\text{red}}^*(G)$ norms on $C_c(G)$ induce the norms

$$\|x\|_{\infty, \text{max}} := \inf\{\|f\|_{\text{max}} \mid f \in \pi^{-1}(x)\}, \quad \|x\|_{\infty, \text{red}} := \inf\{\|f\|_{\text{red}} \mid f \in \pi^{-1}(x)\}$$

on $\mathbb{C}[\Gamma]$, giving rise to completions $C_{\infty, \text{max}}^*(\Gamma)$ and $C_{\infty, \text{red}}^*(\Gamma)$ of $\mathbb{C}[\Gamma]$ respectively. As $*$ -representations of $\mathbb{C}[\Gamma]$ pull back to $*$ -representations of $C_c(G)$ via the map π from line (2), it is immediate that $C_{\infty, \text{max}}^*(\Gamma) = C_{\text{max}}^*(\Gamma)$.

Now, putting all this together, we get a commutative diagram of C^* -algebras

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{n \in \mathbb{N}} C_{\text{red}}^*(\Gamma_n) & \longrightarrow & C_{\text{max}}^*(G) & \longrightarrow & C_{\text{max}}^*(\Gamma) \longrightarrow 0, \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bigoplus_{n \in \mathbb{N}} C_{\text{red}}^*(\Gamma_n) & \longrightarrow & C_{\text{red}}^*(G) & \longrightarrow & C_{\infty, \text{red}}^*(\Gamma) \longrightarrow 0 \end{array} \quad (3)$$

where the vertical maps are the canonical quotients induced by the identity map on $C_c(G)$ and $\mathbb{C}[\Gamma]$, and where the horizontal lines are both exact². The five lemma then gives that $C_{\max}^*(G) = C_{\text{red}}^*(G)$ if and only if $C_{\max}^*(\Gamma) = C_{\infty, \text{red}}^*(\Gamma)$; to complete the proof, we must show therefore show that for any $x \in \mathbb{C}[\Gamma]$ we have

$$\|x\|_{\infty, \text{red}} = \sup_{n \in \mathbb{N} \cup \{\infty\}} \|\lambda_n(x)\|.$$

Fix an element $x \in \mathbb{C}[\Gamma]$. For some suitably large N and all $n \geq N$, the quotient map $\pi_n : \Gamma \rightarrow \Gamma_n$ is injective on the support of x . Define f to be the element of $C_c(G)$ given by

$$f(n, g) = \begin{cases} x(h) & n \geq N \text{ or } n = \infty, \text{ and } g = \pi_n(h) \text{ for } h \in \text{supp}(x) \\ 0 & n < N \end{cases},$$

which is clearly a lift of x for the quotient map $\pi : C_c(G) \rightarrow \mathbb{C}[\Gamma]$. Now, for each $n \in \mathbb{N}$, write f_n for the restriction of f to $\{(m, g) \in G \mid m \geq n \text{ or } m = \infty\}$ and note by consideration of the kernel $C_c(G_{\mathbb{N}})$ of $\pi : C_c(G) \rightarrow \mathbb{C}[\Gamma]$ that we have

$$\|x\|_{\infty, \text{red}} = \limsup_{n \rightarrow \infty} \|f_n\|_{\text{red}}.$$

However, by definition of the reduced norm (Definition 2.5), $\|f_n\|_{\text{red}}$ is just $\sup_{m \geq n \text{ or } m = \infty} \|\rho_m(f)\|$, and by definition of f , this is just $\sup_{m \geq n \text{ or } m = \infty} \|\lambda_m(x)\|$. We conclude that

$$\|x\|_{\infty, \text{red}} = \lim_{n \rightarrow \infty} \sup_{m \geq n \text{ or } m = \infty} \|\lambda_m(x)\|.$$

On the other hand, as the groups K_n are nested, we have that $\|\lambda_n(x)\| \geq \|\lambda_m(x)\|$ whenever $n \geq m$, so

$$\lim_{n \rightarrow \infty} \sup_{m \geq n \text{ or } m = \infty} \|\lambda_m(x)\| = \sup_{n \in \mathbb{N} \cup \{\infty\}} \|\lambda_n(x)\|;$$

this completes the proof. \square

Now, combining Lemmas 2.4 and 2.6, to complete the proof of Theorem 1.2 it suffices to find an approximated group $(\Gamma, (K_n))$ such that Γ is not amenable, but such that the norm on $\mathbb{C}[\Gamma]$ from line (1) equals the maximal norm. We will do this in the next lemma, using property *property FD* of Lubotzky and Shalom [8] as an essential tool.

Lemma 2.7. *Say F_2 is the free group on two generators. Set*

$$K_n = \cap \{ \text{Ker}(\phi) \mid \phi : F_2 \rightarrow \Gamma \text{ a group homomorphism, } |\Gamma| \leq n \}.$$

²Exactness of the top line is a special case of [1, Lemma 6.3.2], but in our case exactness of both lines follows directly from the discussion above.

Then the pair $(F_2, (K_n))$ is an approximated group. Moreover, if $\lambda_n : \mathbb{C}[F_2] \rightarrow \mathcal{B}(l^2(\Gamma_n))$ is as in the statement of Lemma 2.6, then

$$\|x\|_{\max} = \sup_{n \in \mathbb{N} \cup \{\infty\}} \|\lambda_n(x)\|$$

for all $x \in \mathbb{C}[F_2]$.

Proof. Note first that as F_2 is finitely generated and as for each $n \in \mathbb{N}$ there are only finitely many isomorphism types of groups of cardinality at most n , the collection

$$\{\text{Ker}(\phi) \mid \phi : F_2 \rightarrow F \text{ a group homomorphism, } |F| \leq n\}$$

of (finite index, normal) subgroups of F_2 is finite. Hence each K_n is of finite index in F_2 . Moreover, the sequence (K_n) is clearly nested, and each K_n is clearly normal. Finally, note that F_2 is residually finite, whence $\bigcap_n K_n$ is trivial. Hence (K_n) is an approximating sequence for F_2 .

Now, by [8, Theorem 2.2], the collection of all unitary representations of F_2 that factor through a finite quotient is dense in the unitary dual of F_2 . Equivalently (compare the discussion in [8, Section 1]), if we write S for the collection of all irreducible $*$ -representations of $\mathbb{C}[F_2]$ that come from unitary representations of F_2 that factor through finite quotients, then

$$\|x\|_{\max} = \sup\{\|\rho(x)\| \mid \rho \in S\}.$$

However, any quotient map of F_2 with finite image contains all but finitely many of the K_n in its kernel. Using the basic fact from representation theory that any irreducible representation of a finite group is a subrepresentation of the regular representation, we have that any representation in S is a subrepresentation of λ_n for all suitably large n , and are done. \square

As F_2 is non-amenable, this completes the proof of Theorem 1.2. We conclude this section with a few comments about the sort of approximated groups Lemma 2.7 applies to.

Remarks 2.8. 1. For many non-amenable Γ , there are no approximating sequences (K_n) for Γ such that the condition ' $\|x\|_{\max} = \sup_{n \in \mathbb{N} \cup \{\infty\}} \|\lambda_n(x)\|$ for all $x \in \mathbb{C}[\Gamma]$ ' holds. For example, results of Bekka [2] show this for $\Gamma = SL(3, \mathbb{Z})$ and many other higher rank arithmetic groups.

2. On the other hand, there are certainly other non-amenable groups that we could use to replace F_2 in Lemma 2.7: for example, any fundamental group of a closed Riemann surface of genus at least two would work. See [8, Theorem 2.8] for this and other examples.

3. Even for $\Gamma = F_2$, the specific choice of subgroups in Lemma 2.7 is important. For example, say we realise F_2 as a finite index subgroup of $SL(2, \mathbb{Z})$, and choose K_n to be the intersection of the kernel of the reduction map

$SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/2^n\mathbb{Z})$ and F_2 . Then (K_n) is an approximating sequence for F_2 with respect to which F_2 has property (τ) - this follows from Selberg's theorem [12] as in [7, Examples 4.3.3 A]. Property (τ) in this case means that if we set S to be the collection of all unitary representations of F_2 that factor through some quotient F_2/K_n , then the trivial representation is isolated in the closure \bar{S} of S taken with respect to the Fell topology. However, the trivial representation is not isolated in the full unitary dual of F_2 (i.e. F_2 does not have property (T)). Putting this together, the pair $(F_2, (K_n))$ will not satisfy the condition ' $\|x\|_{\max} = \sup_{n \in \mathbb{N} \cup \{\infty\}} \|\lambda_n(x)\|$ for all $x \in \mathbb{C}[\Gamma]$ '.

3 Concluding remarks

We conclude the paper with some (mainly inconclusive!) comments and questions.

Remark 3.1. It is clear from the proof of Lemma 2.6 that failure of the sequence

$$0 \rightarrow C_{\text{red}}^*(G_{\mathbb{N}}) \rightarrow C_{\text{red}}^*(G) \rightarrow C_{\text{red}}^*(\Gamma) \rightarrow 0$$

to be exact, and thus failure of the groupoid G to be exact³ is key for our examples to be possible. Matsumura [9] and Kerr [6, Theorem 2.7] have in fact shown that if $X \rtimes \Gamma$ is an exact transformation groupoid (equivalently, if Γ is an exact group) such that $C(X) \rtimes_{\max} \Gamma = C(X) \rtimes_{\text{red}} \Gamma$, then the groupoid $X \rtimes \Gamma$ is amenable. In the face of our example, it is natural to ask the following questions.

- (i) Can one extend the Kerr-Matsumura result to exact (étale) groupoids?
- (ii) Are there examples of (non-amenable) transformation groups $X \rtimes \Gamma$ such that $C(X) \rtimes_{\max} \Gamma = C(X) \rtimes_{\text{red}} \Gamma$ where Γ is a non-exact group?

As far as we know, no-one has seriously attempted the sort of study of exactness for groupoids that would be a necessary first step in understanding (i), but maybe this would be interesting. Given the current state of knowledge about non-exact groups, (ii) is probably very difficult; our results in this note can be seen as giving a little evidence that such examples might exist, but we do not know enough to speculate either way.

Remark 3.2. The main result of this note makes the following problem very natural: give a groupoid theoretic characterisation of when the maximal and reduced groupoid C^* -algebras are equal.

Remark 3.3. Say X is a bounded geometry⁴ metric space. Associated to X are maximal and reduced *uniform Roe algebras* $C_{u, \max}^*(X)$ and $C_{u, \text{red}}^*(X)$. In coarse geometry, the natural analogue of the question addressed in this note is:

³A group(oid) G is *exact* if taking the reduced crossed product with G takes short exact sequences of G - C^* -algebras to short exact sequences of C^* -algebras.

⁴This means that for each $r > 0$ there is a uniform bound on the cardinality of all r -balls.

is it possible that $C_{u,\max}^*(X) = C_{u,\text{red}}^*(X)$ for some space X without property A? Indeed, associated to X is a coarse groupoid $G(X)$ which is (topologically) amenable if and only if X has property A, and the maximal and reduced uniform Roe algebras of X identify with the maximal and reduced groupoid C^* -algebras of $G(X)$.

This note grew out of an attempt to understand this question (for the specific example looked at in [13, Example 1.15]). We were not, however, able to make any progress on the coarse geometric special case of the general groupoid question.

Remark 3.4. It is also very natural to ask if examples with the sort of properties in Theorem 1.2 are possible for principal⁵ groupoids: the coarse groupoids mentioned above are a special case, as are many other interesting examples coming from equivalence relations and free actions. Again our results seem to shed no light on this question, and we do not know enough to speculate either way.

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⁵This means that the only elements of the groupoid with the same source and range are the units.

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